Singular deformation of domains and several spectral problems

Eigenvalues of the Laplacian in a singularly perturbed domain

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Domain shape :

Solutions of PDE and their structure depend on the geometric properties of domains.

Spectrum :

Notion to discriminate properties of waves.
Spectral problems of PDEs often play important roles.

Physical Background:

Sound waves (music instrument). Resonances through space with obstacles or boundaries.

Heat propagation through complicated shaped space or composite material

Oscillation of buildings (tall building, lattice or box structure)

Pattern formation in Reaction-Diffusion phenomena (or biological phenomena)

Part I : Eigenvalues of the Laplacian in a singularly perturbed domain

Eigenvalue problem

Ω : a bounded domain in $\mathbb{R}^n$ $(n \geqq 2)$ with a smooth boundary $\partial\Omega$

$$(1) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \quad \Phi = 0 \quad \text{on} \quad \partial\Omega \text{ (Dirichlet B.C.)}$$

We consider the equation (1). For each case, (λ, Φ) is an unknown pair of a number and a function. If there is λ with a non-trivial Φ, λ is called an eigenvalue. It is known that there is an infinite discrete sequence of positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ which are arranged in increasing order with counting multiplicities. Denote a corresponding complete system of the orthonormal eigenfunctions by $\{\Phi_k\}_{k=1}^{\infty} \subset L^2(\Omega)$.

(cf. Books of Courant-Hilbert, L.C.Evans, Edmunds-Evans)

Basic Problem–Singular deformation of domains

For a bounded domain $\Omega \subset \mathbb{R}^n$, we deal with a perturbed domain $\Omega(\epsilon)$ ($\epsilon > 0$) of the following two types.

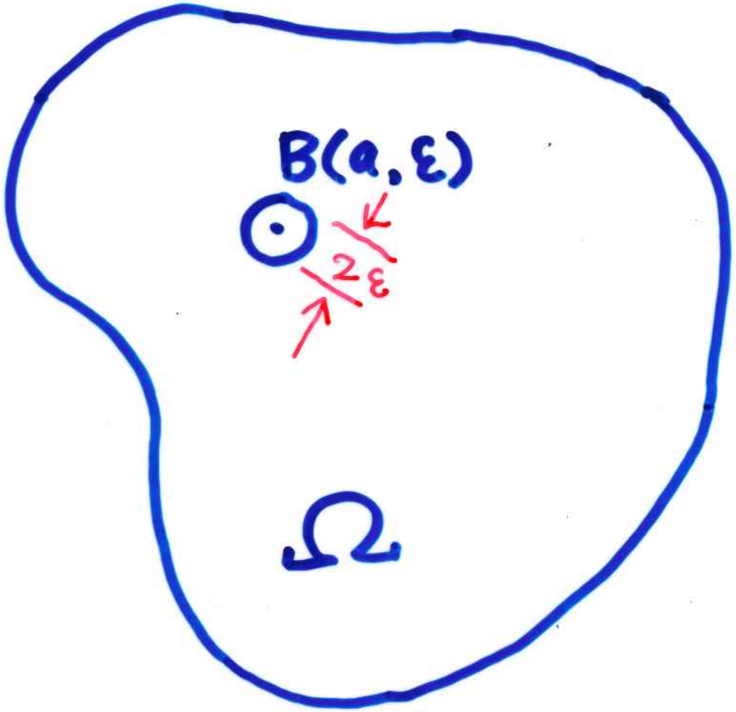
(A) $\Omega(\epsilon)$ has a small hole or a thin defect (tunnel)
(with some B.C. for emerging boundary).
$\Omega(\epsilon)$ increases as $\epsilon \to 0$.

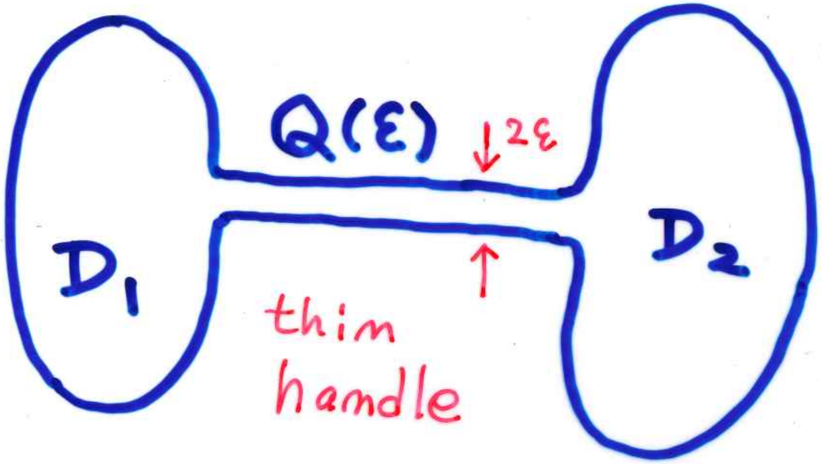
(B) Some portion of $\Omega(\epsilon)$ shrinks to a low dimensional set.
$\Omega(\epsilon)$ decreases as $\epsilon \to 0$.

How does each eigenvalue $\lambda_k(\epsilon)$ of the Laplacian behaves when $\epsilon \to 0$?

See (A) : Swanson ('63'77), Rauch-Taylor('75), Ozawa ('81, '83),..., (B) : Beale ('75), Chavel-Feldman('81), Ram ('85), ... for early works. See Jimbo ('15) and Jimbo-Kosugi ('09) for the details of I-(A) and I-(B) of the lecture.

Two typical cases (2 dimension)





$\Omega(\varepsilon) = \Omega \setminus B(a, \varepsilon)$

Domain with a small hole

$\Omega(\varepsilon) = D_1 \cup D_2 \cup Q(\varepsilon)$

Domain with a thin handle

(A) Domain with a small hole

Let $\boldsymbol{a} \in \Omega$ be a point and

$$\Omega(\epsilon) = \Omega \setminus \overline{B(\boldsymbol{a}, \epsilon)}, \quad \Gamma(\epsilon) = \partial B(\boldsymbol{a}, \epsilon), \quad \Gamma = \partial\Omega.$$

Dirichlet B.C. on $\Gamma(\epsilon)$

$$(2-D) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma(\epsilon) \cup \Gamma$$

Neumann B.C. on $\Gamma(\epsilon)$

$$(2-N) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \text{ on } \Gamma, \quad \partial\Phi/\partial\nu = 0 \text{ on } \Gamma(\epsilon)$$

The $k-$th eigenvalue of (2-D) and (2-N) are denoted by $\lambda_k^D(\epsilon)$ and $\lambda_k^N(\epsilon)$, respectively.

Theorem ($n = 2$ or $n = 3$). Assume λ_k is simple in (1)

$$\lambda_k^D(\epsilon) = \lambda_k + \begin{cases} 4\pi\Phi_k(\boldsymbol{a})^2\epsilon + \text{H.O.T.} & (n = 3) \\ (2\pi/\log(1/\epsilon))\Phi_k(\boldsymbol{a})^2 + \text{H.O.T.} & (n = 2) \end{cases}$$

Theorem ($n = 2$ or $n = 3$). Assume λ_k is simple in (1)

$$\lambda_k^N(\epsilon) = \lambda_k + \begin{cases} \pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + (4\lambda_k/3)\Phi_k(\boldsymbol{a})^2)\epsilon^3 + \text{H.O.T.} & (n = 3) \\ \pi\,(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + \lambda_k\Phi_k(\boldsymbol{a})^2)\epsilon^2 + \text{H.O.T.} & (n = 2) \end{cases}$$

cf. S.Ozawa ('81,'83) for the above results.

Remark. Swanson ('63) gave "some perturbation formula" for $\lambda^D_{k,\epsilon}$, previously.
These results are proved by the method of "Approximate Green function". There are also results for Robin condition on $\Gamma(\epsilon)$ (cf. Ozawa ('83,'92), Roppongi ('93), Ozawa-Roppongi ('92)).

There are many related works in different situations (generalization or elaboration). See Maz'ya-Nazarov-Plamenevsky('85, '00), Flucher('95), Ammari-Kang-Lim-Zribi ('10), Lanza de Christoforis ('12), ...

Proof of Perturbation formula of the eigenvalue $n = 2$, Neumann B.C.

A bounded domain $\Omega \subset \mathbb{R}^2$, a fixed point $\boldsymbol{a} \in \Omega$. The eigenvalue problem

$$(3) \qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \partial\Omega \\ \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \Gamma(\epsilon) = \partial B(\boldsymbol{a}, \epsilon) \end{cases}$$

$\{\lambda_k(\epsilon)\}_{k=1}^{\infty}$: the set of eigenvalues.
$\{\Phi_{k,\epsilon}\}_{k=1}^{\infty}$: system of corresponding eigenfunctions with $(\Phi_{p,\epsilon}, \Phi_{q,\epsilon})_{L^2(\Omega(\epsilon))} = \delta(p,q)$.

$$(4) \qquad \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega, \quad \Phi = 0 \quad \text{on} \quad \partial\Omega$$

$\{\lambda_k\}_{k=1}^{\infty}$: the set of eigenvalues
$\{\Phi_k\}_{k=1}^{\infty}$: system of corresponding eigenfunctions with $(\Phi_p, \Phi_q)_{L^2(\Omega)} = \delta(p,q)$.

We want to closely look at $\lambda_k(\epsilon) - \lambda_k$. There are two parts in the process of proof.

(i) Characterization of the behavior of the true eigenfunction $\Phi_{k,\epsilon}$

(ii) Construction a good approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$

(i) Characterization of $\Phi_{k,\epsilon}$

Proposition. For any sequence of positive numbers $\{\epsilon(p)\}_{p=1}^{\infty}$ with $\lim_{p\to\infty}\epsilon(p)=0$, there exist a subsequence $\{\epsilon(p(q))\}_{q=1}^{\infty}$ and a sequence $\{\lambda'_k\}_{k=1}^{\infty}$ with an complete orthonormal system $\{\Phi'_k\}_{k=1}^{\infty}\subset L^2(\Omega)$ such that

$$\Delta\Phi'_k+\lambda'_k\Phi'_k=0 \text{ in } \Omega, \quad \Phi'_k=0 \text{ on } \partial\Omega,$$

and

$$\lim_{q\to\infty}\lambda_k(\epsilon(p(q)))=\lambda'_k, \quad \lim_{q\to\infty}\sup_{x\in\Omega(\epsilon(p(q)))}|\Phi_{k,\epsilon(p(q))}(x)-\Phi'_k(x)|=0 \quad (\forall k\in\mathbb{N}).$$

We omit the details of the proof.

Estimation of solutions of elliptic equations away from the small hole.

By the aid of "Barrier functions", we prove a uniform bound in $\Omega(\epsilon)$.

Proof of uniform convergence of $\Phi_{k,\epsilon}$ of $\Omega(\epsilon)$.

Proposition. $\lambda_k=\lambda'_k$ $(k\geqq 1)$ and $\lim_{\epsilon\to 0}\lambda_k(\epsilon)=\lambda_k$.

(ii) Construction of approximate eigen function $\widetilde{\Phi}_{k,\epsilon}$. Modify the eigenfunction Φ_k of Ω around the hole $B(\boldsymbol{a},\epsilon)$. Prepare the function

$$\eta_k(x) = \frac{\langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle}{|x - \boldsymbol{a}|^2} \quad (\text{harmonic in} \quad \mathbb{R}^2 \setminus \{\boldsymbol{a}\}).$$

It is easy to calculate

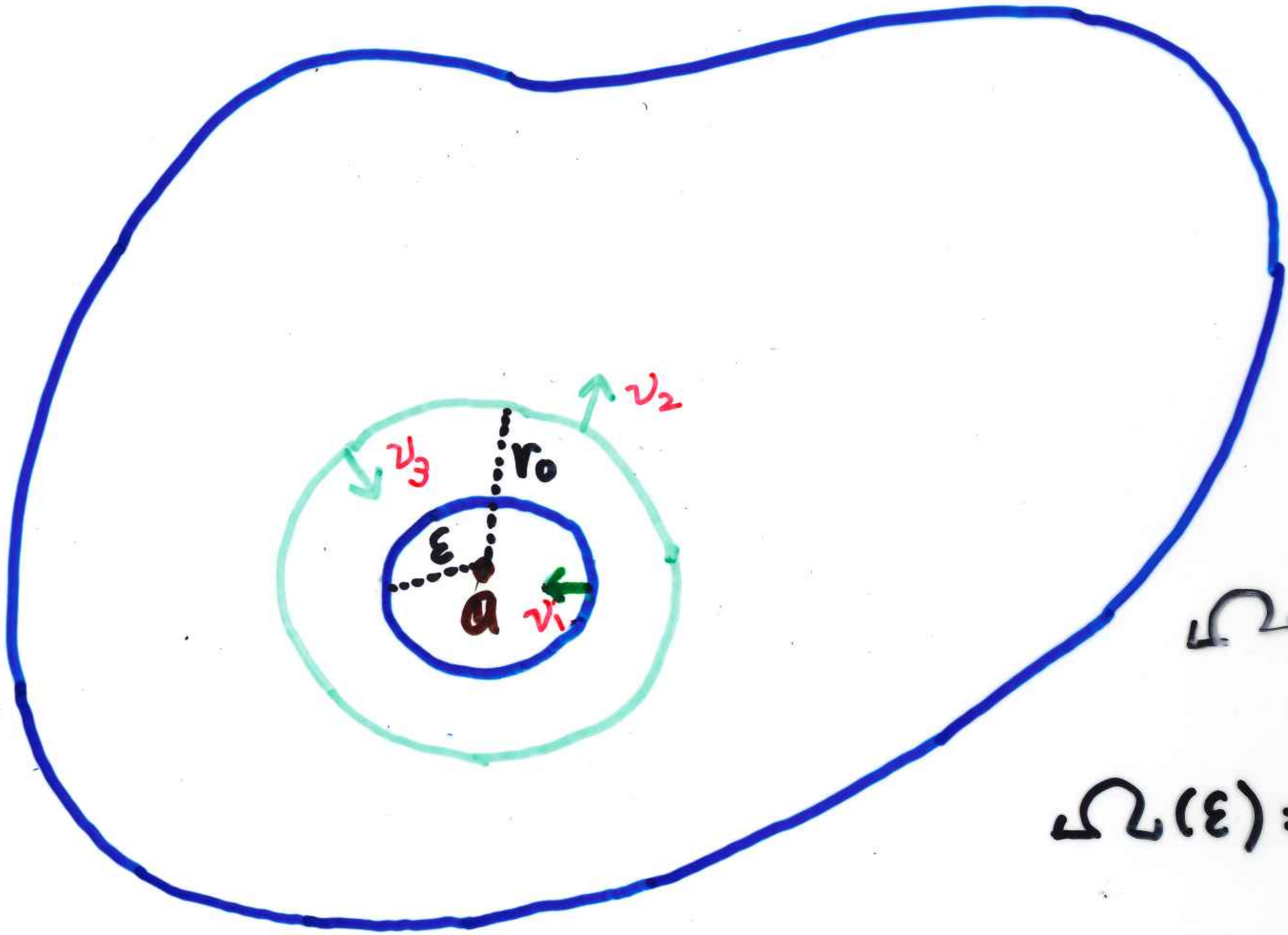
$$\begin{aligned}
\nabla \eta_k(x) &= \frac{\nabla \Phi_k(\boldsymbol{a})}{|x - \boldsymbol{a}|^2} + \langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle \frac{(x - \boldsymbol{a})}{|x - \boldsymbol{a}|} \frac{(-2)}{|x - \boldsymbol{a}|^3} \\
&= \frac{\nabla \Phi_k(\boldsymbol{a})}{|x - \boldsymbol{a}|^2} - 2 \langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle \frac{(x - \boldsymbol{a})}{|x - \boldsymbol{a}|^4} \\
\Delta \eta_k(x) &= \frac{-2 \langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle}{|x - \boldsymbol{a}|^4} - 2 \langle \nabla \Phi_k(\boldsymbol{a}), \frac{x - \boldsymbol{a}}{|x - \boldsymbol{a}|^4} \rangle \\
-2 \langle \nabla \Phi_k(\boldsymbol{a}), x - \boldsymbol{a} \rangle & \left(\frac{2}{|x - \boldsymbol{a}|^4} - \frac{4|x - \boldsymbol{a}|^2}{|x - \boldsymbol{a}|^6} \right) = 0 \qquad (x \neq \boldsymbol{a})
\end{aligned}$$

Put a function $\widetilde{\Phi}_{k,\epsilon}$ as follows

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) + \epsilon^2 \eta_k(x) & (x \in B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon)) \\ \Phi_k(x) + \epsilon^2 \widehat{\eta}_k(x) & (x \in \Omega \setminus B(\boldsymbol{a}, r_0)) \end{cases}$$

where $\widehat{\eta}_k$ is the unique solution $\widehat{\eta}$ of

$$\Delta \widehat{\eta} = 0 \text{ in } \Omega \setminus B(\boldsymbol{a}, r_0), \qquad \widehat{\eta}(x) = 0 \text{ on } \partial\Omega, \qquad \widehat{\eta}(x) = \eta_k(x) \text{ on } \partial B(\boldsymbol{a}, r_0).$$



$$\Omega \subset R^2$$

$$\Omega(\varepsilon) = \Omega \setminus B(a,\varepsilon)$$

Other choice of approximate eigenfunction

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) + \eta_{k,\epsilon}(x) & (x \in B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon)) \\ \Phi_k(x) & (x \in \Omega \setminus B(\boldsymbol{a}, r_0)) \end{cases}$$

where $\eta = \eta_{k,\epsilon} \in C^2(B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon))$ is the unique solution of

$$\Delta \eta = 0 \text{ in } B(\boldsymbol{a}, r_0) \setminus B(\boldsymbol{a}, \epsilon), \quad \eta = 0 \text{ on } \partial B(\boldsymbol{a}, r_0), \ \frac{\partial \eta}{\partial \nu_1} = -\frac{\partial \Phi_k}{\partial \nu_1} \quad \text{on} \quad \partial B(\boldsymbol{a}, \epsilon).$$

Here ν_1 is the inward unit normal vector on $\partial B(\boldsymbol{a}, \epsilon)$.

The equation is written as

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \varphi \rangle - \lambda_k(\epsilon) \Phi_{k,\epsilon} \varphi) dx = 0 \qquad (\forall \varphi \in H^1(\Omega(\epsilon)) \text{ with } \varphi = 0 \text{ on } \partial\Omega)$$

Substitute $\varphi = \widetilde{\Phi}_{k,\epsilon}$, we have

$$\int_{\Omega(\epsilon)} (\langle \nabla \Phi_{k,\epsilon}, \nabla \widetilde{\Phi}_{k,\epsilon} \rangle - \lambda_k(\epsilon) \Phi_{k,\epsilon} \widetilde{\Phi}_{k,\epsilon}) dx = 0$$

Swanson trick : One method to deduce the perturbation of the eigenvalue.

Looking into this integral equality leads us to see the details of $\lambda_k(\epsilon) - \lambda_k$.

$$\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \langle \nabla\Phi_{k,\epsilon}, \nabla(\Phi_k+\epsilon^2\eta_k)\rangle dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \langle \nabla\Phi_{k,\epsilon}, \nabla(\Phi_k+\epsilon^2\widehat{\eta}_k)\rangle dx$$

$$-\lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\widehat{\eta}_k)dx\right) = 0$$

Gauss-Green formula gives

$$\int_{\partial(B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon))} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu}(\Phi_k+\epsilon^2\eta_k)dS - \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\Delta\Phi_k dx$$

$$+\int_{\partial(\Omega\setminus B(\boldsymbol{a},r_0))} \Phi_{k,\epsilon}\frac{\partial}{\partial\nu}(\Phi_k+\epsilon^2\widehat{\eta}_k)dS - \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\Delta\Phi_k dx$$

$$-\lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi_{k,\epsilon}\,(\Phi_k+\epsilon^2\widehat{\eta}_k)dx\right) = 0$$

Using $\Delta\Phi_k = -\lambda_k \Phi_k$ we get

$$(\lambda_k(\epsilon)-\lambda_k)\int_{\Omega(\epsilon)}\Phi_{k,\epsilon}\Phi_k dx = \int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k+\epsilon^2\eta_k)dS + \int_{\partial B(\boldsymbol{a},r_0)}\Phi_{k,\epsilon}\frac{\partial}{\partial\nu_2}(\epsilon^2\eta_k)dS$$

$$+\int_{\partial B(\boldsymbol{a},r_0)}\Phi_{k,\epsilon}\frac{\partial}{\partial\nu_3}(\epsilon^2\eta_k)dS - \lambda_k(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\,(\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)}\Phi_{k,\epsilon}\,(\epsilon^2\widehat{\eta}_k)dx\right)$$

$$= I_1(\epsilon)+I_2(\epsilon)+I_3(\epsilon)+I_4(\epsilon)$$

ν_1 is the unit outward normal vector of $\partial B(\boldsymbol{a},\epsilon)$ at $|x-\boldsymbol{a}|=\epsilon$
ν_2 is the unit outward normal vector of $\partial B(\boldsymbol{a},r_0)$ at $|x-\boldsymbol{a}|=r_0$
ν_3 is the unit outward normal vector of $\partial(\Omega\setminus B(\boldsymbol{a},r_0))$ at $|x-\boldsymbol{a}|=r_0$

Similarly, we get

$$(\lambda_\ell(\epsilon)-\lambda_k)\int_{\Omega(\epsilon)}\Phi_{\ell,\epsilon}\Phi_k dx = \int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k+\epsilon^2\eta_k)dS + \int_{\partial B(\boldsymbol{a},r_0)}\Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_2}(\epsilon^2\eta_k)dS$$

$$+\int_{\partial B(\boldsymbol{a},r_0)}\Phi_{\ell,\epsilon}\frac{\partial}{\partial\nu_3}(\epsilon^2\eta_k)dS - \lambda_\ell(\epsilon)\left(\int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)}\Phi_{\ell,\epsilon}\,(\epsilon^2\eta_k)dx + \int_{\Omega\setminus B(\boldsymbol{a},r_0)}\Phi_{\ell,\epsilon}\,(\epsilon^2\widehat{\eta}_k)dx\right)$$

for any $k,\ell \geqq 1$.

Estimate the right hand side, we can prove

$$(\lambda_\ell(\epsilon) - \lambda_k)(\Phi_{\ell,\epsilon}, \Phi_k)_{L^2(\Omega(\epsilon))} = O(\epsilon^2)$$

(with the aid of calucation)and accordingly , we can also see

$$\lim_{\epsilon \to 0} \lambda_k(\epsilon) = \lambda_k$$

for any $k \geqq 1$.

Evalueate and estimate the terms $I_1(\epsilon), I_2(\epsilon), I_3(\epsilon), I_4(\epsilon)$ of the right hand side.

On $\partial B(\boldsymbol{a},\epsilon)$ (i.e. $|x-\boldsymbol{a}|=\epsilon$), we have

$$\frac{\partial}{\partial \nu_1}(\Phi_k+\epsilon^2\eta_k)=\langle \nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle=O(\epsilon)$$

and we get

$$\frac{1}{\epsilon^2}I_1(\epsilon)=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\frac{\partial}{\partial\nu_1}(\Phi_k+\epsilon^2\eta_k)dS=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_{k,\epsilon}\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}\Phi_k'\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS+\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)}(\Phi_{k,\epsilon}-\Phi_k')\langle\nabla\Phi_k(x)-\nabla\Phi_k(\boldsymbol{a}),\nu_1\rangle dS$$

$$=\widetilde{I}_{1,1}(\epsilon)+\widetilde{I}_{1,2}(\epsilon)=\widetilde{I}_{1,1}(\epsilon)+o(1)$$

For $\epsilon = \epsilon(p(q))$, we have

$$I_4(\epsilon) = -\lambda_k \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Phi'_k \eta_k dx - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

$$= \epsilon^2 \int_{B(\boldsymbol{a},r_0)\setminus B(\boldsymbol{a},\epsilon)} \Delta\Phi'_k \eta_k dx - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

$$= \epsilon^2 \int_{\partial B(\boldsymbol{a},\epsilon)} \frac{\partial \Phi'_k}{\partial \nu_1} \eta_k dS + \epsilon^2 \int_{\partial B(\boldsymbol{a},r_0)} \frac{\partial \Phi'_k}{\partial \nu_2} \eta_k dS$$

$$-\epsilon^2 \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_1} dS - \epsilon^2 \int_{\partial B(\boldsymbol{a},r_0)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_2} dS - \lambda_k \epsilon^2 \int_{\Omega\setminus B(\boldsymbol{a},r_0)} \Phi'_k \widehat{\eta}_k dx + o(\epsilon^2)$$

Comparing the terms of the right hand side with $I_2(\epsilon)$, $I_3(\epsilon)$, we get

$$\frac{1}{\epsilon^2}(I_2(\epsilon)+I_3(\epsilon)+I_4(\epsilon)) = \int_{\partial B(\boldsymbol{a},\epsilon)} \frac{\partial \Phi'_k}{\partial \nu_1} \eta_k dS - \int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k \frac{\partial \eta_k}{\partial \nu_1} dS + o(1) = \widetilde{I}_{2,1}(\epsilon) + \widetilde{I}_{2,2}(\epsilon) + o(1)$$

Lemma.

$$\widetilde{I}_{1,1}(\epsilon) = \lambda_k \pi \Phi'_k(\boldsymbol{a})\Phi_k(\boldsymbol{a}) + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

$$\widetilde{I}_{2,1}(\epsilon) = -\pi\langle \nabla\Phi'_k(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0),$$

$$\widetilde{I}_{2,2}(\epsilon) = -\pi\langle \nabla\Phi'_k(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + o(1) \quad (\epsilon = \epsilon(p(q)) \to 0).$$

(Sketch of the roof) Evaluate each quantity by Taylor expansiion.

$$\begin{aligned}\widetilde{I}_{1,1} &= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k \langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS \\ &= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi'_k(\boldsymbol{a}) \langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS \\ &+ \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} (\Phi'_k(x) - \Phi'_k(\boldsymbol{a})) \langle \nabla\Phi_k(x) - \nabla\Phi_k(\boldsymbol{a}), \nu_1\rangle dS\end{aligned}$$

$$= \frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'(\boldsymbol{a}) \sum_{\ell=1}^{2}(x_\ell - a_\ell)\langle \frac{\partial}{\partial x_\ell}(\nabla \Phi_k)(\boldsymbol{a}) + O(\epsilon), (-1)\frac{(x-\boldsymbol{a})}{|x-\boldsymbol{a}|}\rangle dS + o(1)$$

$$= -\frac{1}{\epsilon^2}\int_{\partial B(\boldsymbol{a},\epsilon)} \Phi_k'(\boldsymbol{a}) \sum_{\ell_1,\ell_2=1}^{2} \frac{\partial^2 \Phi_k'}{\partial x_{\ell_1}\partial x_{\ell_2}}(\boldsymbol{a})(x_{\ell_1} - a_{\ell_1})(x_{\ell_2} - a_{\ell_2}) dS + o(1)$$

$$= -\pi \Phi_k'(\boldsymbol{a})\Delta\Phi_k(\boldsymbol{a}) + o(1) = \pi\lambda_k \Phi_k'(\boldsymbol{a})\Phi_k(\boldsymbol{a}) + o(1)$$

The remaining two terms $\widetilde{I}_{2,1}(\epsilon)$, $\widetilde{I}_{2,2}(\epsilon)$ are evaluated similarly with the aid of Taylor expansion. □

Eventually we have

$$\lim_{q\to\infty} \frac{\lambda_k(\epsilon(p(q))) - \lambda_k}{\epsilon(p(q))^2} (\Phi_{k,\epsilon(p(q))}, \Phi_k)_{L^2(\Omega(\epsilon(p(q))))}$$
$$= \pi(-2\langle \nabla\Phi'_k(\boldsymbol{a}), \nabla\Phi_k(\boldsymbol{a})\rangle + \lambda_k\pi\Phi'_k(\boldsymbol{a})\Phi_k(\boldsymbol{a}))$$

Since λ_k is simple, we accordingly have $\Phi'_k = \Phi_k$ or $\Phi'_k = -\Phi_k$ and the $\{\epsilon(p)\}_{p=1}^{\infty}$ is arbitrary and conclude

$$\lim_{\epsilon\to 0} \frac{\lambda_k(\epsilon) - \lambda_k}{\epsilon^2} = \pi(-2|\nabla\Phi_k(\boldsymbol{a})|^2 + \lambda_k\pi\Phi_k(\boldsymbol{a})^2)$$

Domain with a thin tubular hole

Let M be a $m-$dimensional smooth compact orientable manifold such that $M \subset \Omega$ and $0 \leqq m \leqq n-2$ and put

$$B(M,\epsilon) = \{x \in \mathbb{R}^n \mid \text{dist}(x,M) < \epsilon\}, \qquad \Gamma = \partial\Omega, \quad \Gamma(M,\epsilon) = \partial B(M,\epsilon).$$

Note $|B(M,\epsilon)| = O(\epsilon^{n-m})$.

Let $\Omega(\epsilon) = \Omega \setminus \overline{B(M,\epsilon)}$ and $\lambda_k^D(\epsilon)$ be the $k-$th eigenvalue of the Laplacian in $\Omega(\epsilon)$ with the Dirichlet B.C. on $\partial\Omega(M,\epsilon)$.

Due to **G.Besson ('85), I.Chavel-D.Feldman ('88), C.Courtois ('95),** the following results have been established.

Theorem. Assume λ_k is simple in (1)

$$\lambda_k^D(\epsilon) - \lambda_k = \begin{cases} \left((n-m-2)|S^{n-m-1}| \int_M \Phi_k(\xi)^2 ds\right) \epsilon^{n-m-2} + \text{H.O.T.} & \text{for } n-m \geqq 3 \\ \left(2\pi \int_M \Phi_k(\xi)^2 ds\right) / \log(1/\epsilon) + \text{H.O.T.} & \text{for } n-m = 2 \end{cases}$$

Here S^{n-m-1} is the unit sphere in $\mathbb{R}^{n-m}$ and "H.O.T." implies "a higher order term".

In case of multiple eigenvalue λ_k

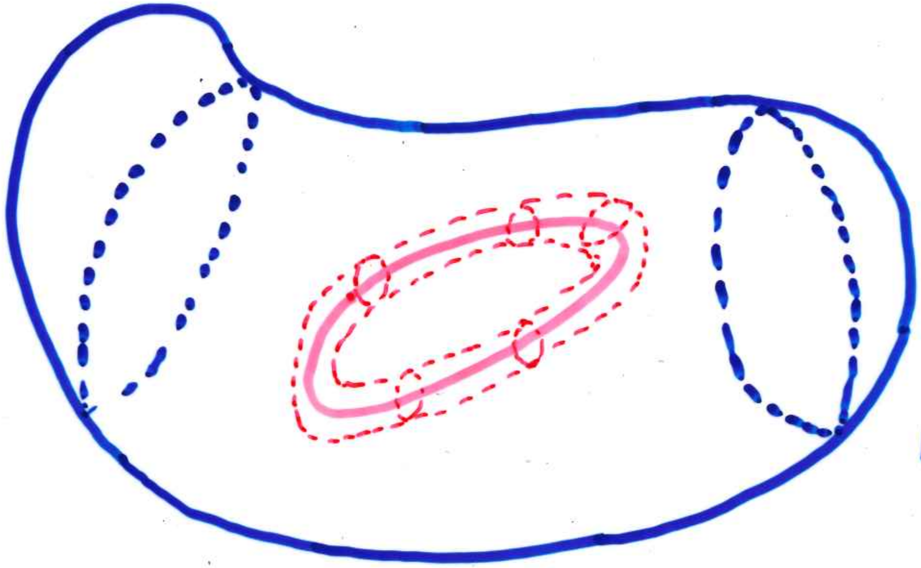
$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+p-1} < \lambda_{k+p}$$

Let $\rho_k \leqq \rho_{k+1} \leqq \cdots \leqq \rho_{k+p-1}$ be the eigenvalue of the matrix

$$\left(\int_M \Phi_i \Phi_j \, dS(\xi) \right)_{k \leqq i,j \leqq k+p-1} .$$

$$\lambda_j^D(\varepsilon) - \lambda_j = \begin{cases} (n-m-2)|S^{n-m-1}| \rho_j \, \varepsilon^{n-m-2} + \text{H.O.T} & (n-m \geqq 3) \\ 2\pi \rho_j / \log(1/\varepsilon) + \text{H.O.T} & (n-m=2) \end{cases}$$

$\Omega \subset \mathbb{R}^3$



$M =$ 

circle in Ω

$B(M, \varepsilon)$: ε-neighborhood of M

$$\Omega(\varepsilon) = \Omega \setminus B(M, \varepsilon)$$

The case of Neumann B.C., Robin B.C. on $\Gamma(M,\epsilon)$

Perturbed eigenvalue problems

$$(5)\qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \qquad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \dfrac{\partial \Phi}{\partial \nu} + \sigma\epsilon^{\tau}\Phi = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \qquad (\text{<= Robin B.C.}) \end{cases}$$

$$(6)\qquad \begin{cases} \Delta\Phi + \lambda\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \quad \Phi = 0 \quad \text{on} \quad \Gamma, \\ \dfrac{\partial \Phi}{\partial \nu} = 0 \quad \text{on} \quad \Gamma(M,\epsilon). \qquad (\text{<= Neumann B.C.}) \end{cases}$$

Here ν is the unit outward normal vector on $\partial\Omega(\epsilon)$ and $\sigma > 0, \tau \in \mathbb{R}$ are parameters.

Eigenvalues and Eigenfunctions in $\Omega(\epsilon)$

Definition. We denote the eigenvalues of (3) by $\{\lambda_k^R(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system by $\{\Phi_{k,\epsilon}^R\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^R, \Phi_{\ell,\epsilon}^R)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Definition. We denote the eigenvalues of (4) by $\{\lambda_k^N(\epsilon)\}_{k=1}^\infty$ and the corresponding complete orthonormal system $\{\Phi_{k,\epsilon}^N\}_{k=1}^\infty \subset L^2(\Omega(\epsilon))$, respectively.

$$(\Phi_{k,\epsilon}^N, \Phi_{\ell,\epsilon}^N)_{L^2(\Omega(\epsilon))} = \delta(k,\ell) \quad (k,\ell \geqq 1).$$

Proposition. For $k \in \mathbb{N}$, $\lambda_k^N(\epsilon) \leqq \lambda_k^R(\epsilon) \leqq \lambda_k^D(\epsilon) \leqq \lambda_k + o(1)$ for $\epsilon \to 0$.

(Sketch of the proof) This is proved by a (rough) test functions

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x)\frac{\log(\epsilon/r)}{\log(\epsilon/r_0)} & \text{for} \quad x \in B(M, r_0) \setminus B(M, \epsilon), r = \mathrm{dist}(x, M) \end{cases}.$$

with the max-min principle through the Rayleigh quotient

$$\mathcal{R}_\epsilon(\Phi) = \|\nabla\Phi\|^2_{L^2(\Omega(\epsilon))}/\|\Phi\|^2_{L^2(\Omega(\epsilon))}$$

$$\lambda_k^D(\epsilon) = \sup_{\dim E \leqq k-1} \inf\{\mathcal{R}_\epsilon(\Phi) \mid \Phi \in H_0^1(\Omega(\epsilon)),\ \Phi \perp E \text{ in } L^2(\Omega(\epsilon))\}$$

Here E is a subspace of $L^2(\Omega(\epsilon))$.

$$\|\widetilde{\Phi}_{k,\epsilon}-\Phi_k\|^2_{L^2(\Omega(\epsilon))}=O(1/|\log\epsilon|^2),\quad \|\nabla(\widetilde{\Phi}_{k,\epsilon}-\Phi_k)\|^2_{L^2(\Omega(\epsilon))}=\begin{cases}O(1/|\log\epsilon|^2) & \text{if}\quad q\geqq 3\\ O(1/|\log\epsilon|) & \text{if}\quad q=2\end{cases}$$

$$(\widetilde{\Phi}_{k,\epsilon},\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\delta(k,k')+O(\frac{1}{|\log\epsilon|}),$$

$$(\nabla\widetilde{\Phi}_{k,\epsilon},\nabla\widetilde{\Phi}_{k',\epsilon})_{L^2(\Omega(\epsilon))}=\lambda_\ell\,\delta(k,k')+\begin{cases}O(\frac{1}{|\log\epsilon|^{1/2}}) & \text{for } q=2\\ O(\frac{1}{|\log\epsilon|}) & \text{for } q\geqq 3\end{cases}$$

Put $F=L.H.[\widetilde{\Phi}_{1,\epsilon},\widetilde{\Phi}_{2,\epsilon},\cdots,\widetilde{\Phi}_{k,\epsilon}]$ and see $\dim(F)=k$.
Take any subspace $E\subset L^2(\Omega(\epsilon))$ with $\dim(E)\leqq k-1$, then there exists

$$\Psi=\sum_{\ell=1}^{k}c_k\widetilde{\Phi}_{\ell,\epsilon}\in F,\quad \Psi\perp E \text{ in } L^2(\Omega(\epsilon)),\quad \sum_{\ell=1}^{k}c_\ell^2=1.$$

Then we have

$$\inf_{\Phi\in H^1_0(\Omega(\epsilon)),\Phi\perp E\,\mathrm{in}\,L^2(\Omega(\epsilon))}\mathcal{R}_\epsilon(\Phi)\leqq\mathcal{R}_\epsilon(\Psi)=\frac{\|\nabla(\sum_{\ell=1}^k c_k\widetilde{\Phi}_{\ell,\epsilon})\|^2_{L^2(\Omega(\epsilon))}}{\|\sum_{\ell=1}^k c_k\widetilde{\Phi}_{\ell,\epsilon}\|^2_{L^2(\Omega(\epsilon))}}$$

$$= \frac{\sum_{1 \leqq \ell, \ell' \leqq k} (\nabla \widetilde{\Phi}_{\ell,\epsilon}, \nabla \widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))} c_\ell c_{\ell'}}{\sum_{1 \leqq \ell, \ell' \leqq k} (\widetilde{\Phi}_{\ell,\epsilon}, \widetilde{\Phi}_{\ell',\epsilon})_{L^2(\Omega(\epsilon))} c_\ell c_{\ell'}} = \frac{\sum_{\ell=1}^k \lambda_\ell (1 + o(1)) c_\ell^2 + \sum_{1 \leqq \ell \neq \ell' \leqq k} o(1) c_\ell c_{\ell'}}{\sum_{\ell=1}^k (1 + o(1)) c_\ell^2 + \sum_{1 \leqq \ell \neq \ell' \leqq k} o(1) c_\ell c_{\ell'}}$$

$$\leqq \frac{\lambda_k + o(1)}{1 - k^2 o(1)} \leqq \lambda_k + o(1)$$

Note that the right hand side is independent of choice of E. Taking sup for all choices of $E \subset L^2(\Omega(\epsilon)), \mathrm{dim} E \leqq k - 1$ with the max min principle

$$\lambda_k^D(\epsilon) \leqq \lambda_k + o(1)$$

Since $\lambda_k \leqq \lambda_k^D(\epsilon)$, $\lim_{\epsilon \to 0} \lambda_k^D(\epsilon) = \lambda_k$ follows.

Proposition (Convergence). For $k \in \mathbb{N}$, we have
$$\lim_{\epsilon\to 0} \lambda_k^R(\epsilon) = \lambda_k \quad (k \in \mathbb{N}), \quad \lim_{\epsilon\to 0} \lambda_k^N(\epsilon) = \lambda_k \quad (k \in \mathbb{N}).$$

Proposition (Uniform bound). For each $k \in \mathbb{N}$, there exist $\epsilon_0 > 0$ and $c(k) > 0$ such that
$$|\Phi_{k,\epsilon}^R(x)| \leqq c(k), \quad |\Phi_{k,\epsilon}^N(x)| \leqq c(k) \quad (x \in \Omega(\epsilon),\ 0 < \epsilon \leqq \epsilon_0).$$

Notation

∇ : the gradient in $\mathbb{R}^n$
∇_M : the tangential gradient on M
∇_N : the normal gradient at a point of the manifold M

$$\nabla\phi = \nabla_M\phi + \nabla_N\phi \quad \text{on} \quad M$$

Notation

Denote the **mean curvature vector** field on M by H. H is a normal vector field on M. As an operator, for a function ϕ defined in a neighborhood of M, H acts on ϕ as a differential in H direction as follows

$$H[\phi](\xi) = \lim_{t\to 0}(\phi(\xi + tH(\xi)) - \phi(\xi))/t \quad \text{at each} \quad \xi \in M.$$

Theorem. Assume that $n-m=q \geqq 3$ and λ_k is simple in (1).
(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_k H[\Phi_k]\right\}ds(\xi).$$

(i) Assume $\tau>1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_k H[\Phi_k]\right\}ds(\xi).$$

(ii) Assume $\tau=1$, then we have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^q}=\frac{|S^{q-1}|}{q}\int_M\left\{-\frac{q}{q-1}|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+(\lambda_k+q\sigma)\Phi_k^2-\Phi_k H[\Phi_k]\right\}ds(\xi).$$

(iii) Assume $-1 < \tau < 1$, then we have
$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^{q+\tau-1}} = \sigma|S^{q-1}|\int_M \Phi_k(\xi)^2 ds(\xi).$$
(iv) Assume $\tau = -1$, then we have
$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^{q-2}} = \frac{\sigma(q-2)}{q-2+\sigma}|S^{q-1}|\int_M \Phi_k(\xi)^2 ds(\xi).$$
(v) Assume $\tau < -1$, then we have
$$\lim_{\epsilon\to 0}\frac{\lambda_k^R(\epsilon)-\lambda_k}{\epsilon^{q-2}} = (q-2)|S^{q-1}|\int_M \Phi_k(\xi)^2 ds(\xi).$$
Here $|S^{q-1}| = 2\pi^{q/2}/\mathbf{\Gamma}(q/2)$, which is is the measure of S^{q-1} and $\mathbf{\Gamma}(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ is the Gamma function.

Theorem. Assume that $n-m=q=2$ and λ_k is simple in (1).
(0) We have

$$\lim_{\epsilon\to 0}\frac{\lambda_k^N(\epsilon)-\lambda_k}{\epsilon^2}=\pi\int_M\left(-2|\nabla_N\Phi_k|^2-|\nabla_M\Phi_k|^2+\lambda_k\Phi_k^2-\Phi_k H[\Phi_k]\right)ds(\xi).$$

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(iii) Assume $-1 < \tau < 1$, then we have

$$\lim_{\epsilon \to 0} \frac{\lambda_k^R(\epsilon) - \lambda_k}{\epsilon^{1+\tau}} = 2\pi\sigma \int_M \Phi_k(\xi)^2 ds(\xi).$$

(iv) Assume $\tau \leqq -1$, then we have

$$\lim_{\epsilon \to 0} (\lambda_k^R(\epsilon) - \lambda_k) \log(1/\epsilon) = 2\pi \int_M \Phi_k(\xi)^2 ds(\xi).$$

The above theorems are in S.Jimbo, Eigenvalues of the Laplacian in a domain with a thin tubular hole, J. Elliptic, Parabolic, Equations **1** (2015).

Remark. It should be noted that in the case $\tau < -1$ in Theorem 3 and Theorem 4, the formula takes the same form as $\lambda_k^D(\epsilon)$ (the case of the Dirichlet B.C. on $\Gamma(M, \epsilon)$). In this case the Robin B.C. is close to the Dirichlet B.C. On the other hand, the formulas for $\lambda_k^R(\epsilon)$ for $\tau > 1$ (in (i)) takes the same form as $\lambda_k^N(\epsilon)$ (in (0)).

Remark. S. Ozawa dealt with $n = 3$, $\dim M = 1$ and proved (iii) in Theorem 4 with other method in his preprint: S. Ozawa, Spectra of the Laplacian and singular variation of domain - removing an $\epsilon-$ neighborhood of a curve, unpublished note (1998).

Sketch of the proof

[I] Characterization of the eigenfunction $\Phi^R_{k,\epsilon}(x)$, $\Phi^N_{k,\epsilon}(x)$

Estimates for uniform bound and convergence

[II] Construction of the approximate eigenfunction $\widetilde{\Phi}^R_{k,\epsilon}(x)$, $\widetilde{\Phi}^N_{k,\epsilon}(x)$

Explicit expression of the approximation

[**Coordinate system in** $B(M, r_0)$]

M : a compact $m-$dimensional smooth manifold (M has a finite covering)

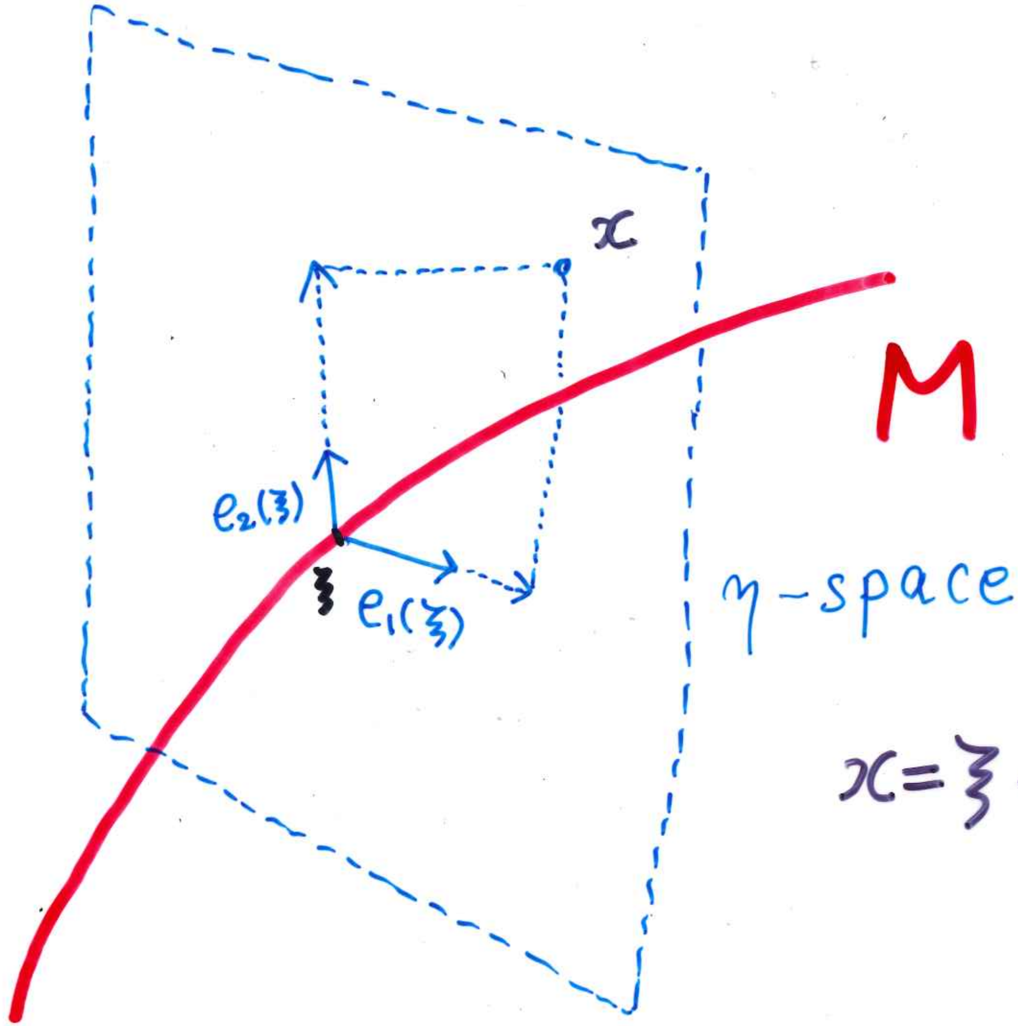
$$\mathbb{R}^n = T_\xi M \oplus N_\xi M \quad (\xi \in M) \qquad \text{(orthogonal decomposition)}$$

Here $\dim(T_\xi M) = m$, $\dim(N_\xi M) = q$.

Let $(e_1(\xi), e_2(\xi), \cdots, e_q(\xi))$ be an orthonormal frame in $N_\xi M$ (smooth in ξ) in a chart of the covering. $\exists r_0 > 0$ such that

$$B(M, r_0) \ni x = \xi + \sum_{\ell=1}^{q} \eta_\ell \, e_\ell(\xi).$$

Denote the second term by $\eta \cdot e(\xi)$.



$$x = \xi + \underbrace{\eta_1 e_1(\xi) + \eta_2 e_2(\xi)}_{\eta \cdot e(\xi)}$$

[**Mean curvature operator (vector) on** M]

The second fundamental form $h_\xi(X,Y)$ of M is defined by the following formula

$$\nabla_Y X = \nabla_Y^M X + h_\xi(X,Y) \in T_\xi M \oplus N_\xi M \quad \text{(orthogonal decomposition)}$$

for any C^1 vector fields X, Y which are defined in a neighborhood of M and tangent to M. The mean curvature vector H of M is defined by

$$H_\xi = \sum_{i=1}^{m} h_\xi(E_i, E_i)$$

for each $\xi \in M$. Here $\{E_1, E_2, \cdots, E_m\}$ is an orthonormal frame of $T_\xi M$ (cf. Kobayashi-Nomizu ('63)).

Lemma. In this coordinate system $(\xi, \eta) = (\xi_1, \cdots, \xi_m, \eta_1, \cdots, \eta_q)$, the mean curvature operator of M is expressed as follows.

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,0)}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{|M} \frac{\partial}{\partial \eta_\ell} = -\sum_{\ell=1}^{q} \sum_{1 \leqq i,j \leqq m} \frac{g^{ij}(\xi,0)}{2} \frac{\partial g_{ij}}{\partial \eta_\ell}(\xi,0) \frac{\partial}{\partial \eta_\ell}$$

It is also expressed as a normal vector field

$$H_\xi = -\sum_{\ell=1}^{q} \frac{1}{\sqrt{g(\xi,\mathbf{0})}} \left(\frac{\partial \sqrt{g(\xi,\eta)}}{\partial \eta_\ell} \right)_{\eta=\mathbf{0}} e_\ell(\xi).$$

Proposition. For a C^2 function u which is defined in $B(M, r_0)$, we have

$$(\Delta u)_{|M} = \Delta_M(u_{|M}) - H[u] + \sum_{\ell=1}^{q} \left(\frac{\partial^2 u}{\partial \eta_\ell^2} \right)_{|\eta=\mathbf{0}} \quad \text{on} \quad M.$$

[I] Uniform bound for the eigenfunction $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

Lemma (Barrier function). There exsits a function $\psi_\epsilon(x)$ (defined from K_1) satisfies the following properties. For any $m_2 > 0$, there exist $\epsilon_1 > 0$, $r_1 \in (0, r_0]$ and $\epsilon_1 > 0$ such that

$$\Delta\psi_\epsilon + m_2\psi_\epsilon \leqq 0 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

$$\frac{\partial\psi_\epsilon}{\partial\nu} \geqq 0 \quad \text{on} \quad \Gamma(M, \epsilon), \quad 1 \leqq \psi_\epsilon(x) \leqq 3 \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon),$$

for any $\epsilon \in (0, \epsilon_1)$.

Estimates for $\Phi^R_{k,\epsilon}$, $\Phi^N_{k,\epsilon}$

For any $r_1 > 0$, there exists $c > 0$ such that $|\Phi^R_{k,\epsilon}(x)| \leqq c$ in $\Omega \setminus B(M, r_1)$ and $0 < \epsilon \leqq r_1/2$ (Elliptic estimates).

By the comparison argument, we have

$$-c\,\psi_\epsilon(x) \leqq \Phi^R_{k,\epsilon}(x) \leqq c\,\psi_\epsilon(x) \quad \text{in} \quad B(M, r_1) \setminus B(M, \epsilon).$$

for $\epsilon > 0$.

Same argument applies to $\Phi^N_{k,\epsilon}$.

[II] Approximate eigenfunction $\widetilde{\Phi}^R_{k,\epsilon}$

We first construct an approximate eigenfunction $\widetilde{\Phi}_{k,\epsilon}$, by modifying Φ_k around M according to the Robin B.C. on $\Gamma(M,\epsilon)$. We consider $\phi(\eta)=\phi(\eta_1,\cdots,\eta_q)$ satisfying

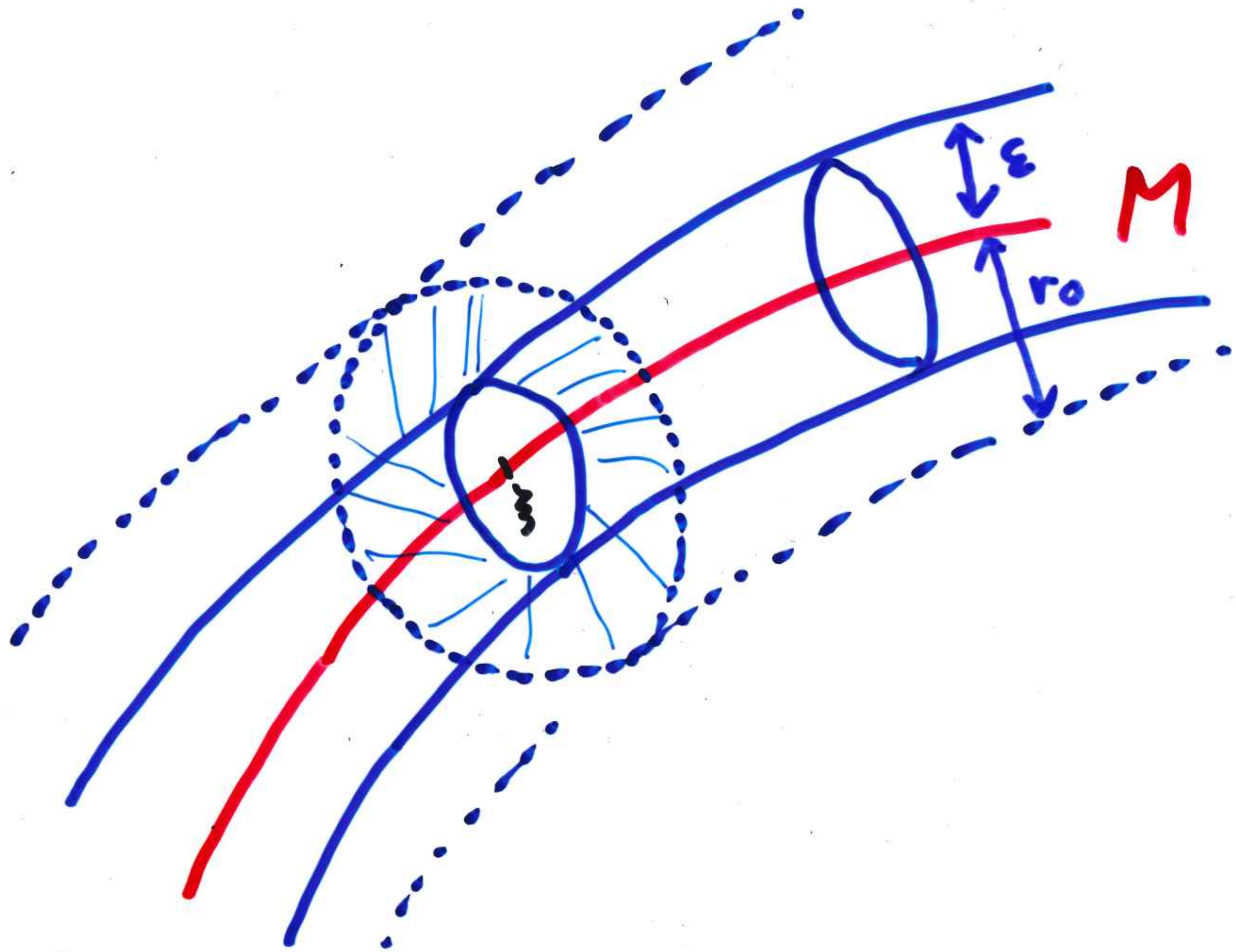
$$\begin{cases} \Delta_\eta \phi = 0 \quad \text{for} \quad \epsilon < |\eta| < r_0, \quad \phi = 0 \quad \text{for} \quad |\eta| = r_0, \\ \left(\dfrac{\partial \phi}{\partial \nu_\eta} + \sigma \epsilon^\tau \phi \right)_{|\eta|=\epsilon} = \left(\dfrac{\partial}{\partial \nu_\eta} \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi)) + \sigma \epsilon^\tau \Phi_k(\xi + \eta \cdot \boldsymbol{e}(\xi)) \right)_{|\eta|=\epsilon} \end{cases}$$

for each $\xi \in M$. Here $\Delta_\eta = \partial^2/\partial\eta_1^2 + \cdots + \partial^2/\partial\eta_q^2$. Basic harmonic functions in η space solutions are given by

$$r^\ell \varphi_{\ell,p}(\omega), \quad r^{-\ell-q+2}\varphi_{\ell,p}(\omega) \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)) \quad \text{harmonic functions in} \quad \mathbb{R}^q \setminus \{\mathbf{0}\}.$$

where $\{\varphi_{\ell,p}(\omega)\}_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)}$ are eigenfunctions of the Laplace-Beltrami operator in S^{q-1}. The eigenvalues $\gamma(\ell)$ and its multiplicity $\iota(\ell)$ are given as follows

$$\gamma(\ell) = \ell(\ell+q-2), \ \ \iota(\ell) = \frac{(2\ell+q-2)(q+\ell-3)!}{(q-2)!\ell!} \quad (\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)$$


ε
M
r_0

The solution of the Laplace equation

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{l,p} r^{\ell} + b_{\ell,p} r^{-\ell-q+2}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

The coefficients $a_{\ell,p}$, $b_{\ell,p}$ can be calculated by the infinite series of relations determined by the boundary condition. From the boundary condition on $|\eta| = r_0$, we have

$$\sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} (a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2}) \varphi_{\ell,p}(\omega) = 0 \quad (\omega \in S^{q-1})$$

which gives

$$a_{\ell,p} r_0^{\ell} + b_{\ell,p} r_0^{-\ell-q+2} = 0 \quad \text{for} \quad \ell \geqq 0, 1 \leqq p \leqq \iota(\ell).$$

ϕ is written by

$$\phi(\eta) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} (r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell}) \varphi_{\ell,p}(\omega) \quad (\epsilon < r < r_0, \omega \in S^{q-1}).$$

We calculate the Robin condition on $|\eta| = \epsilon$. Noting

$$\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial r} = -\sum_{i=1}^{q} \frac{\eta_i}{|\eta|}\frac{\partial}{\partial \eta_i} \quad \text{on} \quad \Gamma(M,\epsilon) = \{x = \xi + \eta \cdot e(\xi) \mid \xi \in M, |\eta| = \epsilon\}$$

we have the equations for the coefficients $a_{\ell,p}, b_{\ell,p}$ as follows.

$$- \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} \left((-\ell - q + 2) r^{-\ell-q+1} - \ell r_0^{-2\ell-q+2} r^{\ell-1} \right)_{r=\epsilon} \varphi_{\ell,p}(\omega)$$

$$+ \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p} \sigma \epsilon^{\tau} \left(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell} \right)_{r=\epsilon} \varphi_{\ell,p}(\omega)$$

$$= -\sum_{i=1}^{q} \langle \nabla \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)), e_i(\xi) \rangle \eta_i / |\eta| + \sigma \epsilon^{\tau} \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi))$$

for $\omega \in S^{q-1}$. Multiply both sides by $\varphi_{p,\ell}$ and integrate on S^{q-1} and we get

$$b_{\ell,p} \left\{ (\ell + q - 2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2} \epsilon^{\ell-1} + \sigma (\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2} \epsilon^{\ell+\tau}) \right\}$$

$$= \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{ (\nabla \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)), e_i(\xi)) \omega_i \} + \sigma \epsilon^{\tau} \Phi_k(\xi + (\epsilon\omega) \cdot e(\xi)) \right\} \varphi_{\ell,p}(\omega) d\omega$$

We used $\omega_i = \eta_i/|\eta|$.

From these equations we get $a_{\ell,p}, b_{\ell,p}$ as follows

$$a_{\ell,p} = -r_0^{-2\ell-q+2} b_{\ell,p}$$

$$b_{\ell,p} = \frac{1}{(\ell+q-2)\epsilon^{-\ell-q+1} + \ell r_0^{-2\ell-q+2}\epsilon^{\ell-1} + \sigma(\epsilon^{-\ell-q+2+\tau} - r_0^{-2\ell-q+2}\epsilon^{\ell+\tau})}$$

$$\times \int_{S^{q-1}} \left\{ -\sum_{i=1}^{q} \{(\nabla\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)), e_i(\xi))\omega_i\} + \sigma\epsilon^{\tau}\Phi_k(\xi + (\epsilon\omega)\cdot e(\xi)) \right\} \varphi_{\ell,p}(\omega)d\omega$$

We remark that these ($\epsilon-$dependent) coefficients $a_{\ell,p}$, $b_{\ell,p}$ are smoothly dependent on $\xi \in M$ since Φ_k is smooth. So we denote this function $\phi(x)$ in $B(M, r_0) \setminus B(M, \epsilon)$ by $G_{k,\epsilon}(x)$. That is

$$G_{k,\epsilon}(x) = \sum_{\ell \geqq 0, 1 \leqq p \leqq \iota(\ell)} b_{\ell,p}(r^{-\ell-q+2} - r_0^{-2\ell-q+2} r^{\ell})\varphi_{\ell,p}(\omega) \quad (x = \xi + (r\omega)\cdot e(\xi)).$$

Definition. The approximate eigenfunction is defined by

$$\widetilde{\Phi}_{k,\epsilon}(x) = \begin{cases} \Phi_k(x) & \text{for} \quad x \in \Omega \setminus B(M, r_0) \\ \Phi_k(x) - G_{k,\epsilon}(x) & \text{for} \quad x = \xi + \eta\cdot e(\xi) \in B(M, r_0) \setminus B(M, \epsilon) \end{cases}$$

$$x = \xi + \eta \cdot e(\xi) \in B(M, r_0) \setminus B(M, \varepsilon)$$

$$G^{(1)}_{k,\varepsilon}(x) = b_{0,1}(\xi)\left(\frac{1}{|\eta|^{q-2}} - \frac{1}{r_0^{q-2}}\right)\frac{1}{|\zeta^{q-1}|^{\frac{1}{2}}}$$

$$+ \sum_{p=1}^{q} b_{1,p}(\xi)\left(\frac{1}{|\eta|^{q-1}} - \frac{|\eta|}{r_0^{q-1}}\right)\frac{q^{\frac{1}{2}}}{|\zeta^{q-1}|^{\frac{1}{2}}}\,\frac{\eta_p}{|\eta|}$$

Lemma. (i) $\ell = 0$

$$b_{0,1} = |S^{q-1}|^{1/2} \begin{cases} \frac{-\epsilon^q}{q(q-2)}\{\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + O(\epsilon)\} & (\tau > 1) \\ \frac{\epsilon^q}{q-2}\{(-1/q)\sum_{i=1}^q \langle e_i(\xi), \nabla^2\Phi_k(\xi)e_i(\xi)\rangle + \sigma\Phi_k(\xi) + O(\epsilon)\} & (\tau = 1) \\ \frac{\sigma\epsilon^{q-1+\tau}}{q-2}(\Phi_k(\xi) + O(\epsilon)) & (-1 < \tau < 1) \\ \frac{\sigma\epsilon^{q-2}}{q-2+\sigma}(\Phi_k(\xi) + O(\epsilon)) & (\tau = -1) \\ \epsilon^{q-2}(\Phi_k(\xi) + O(\epsilon)) & (\tau < -1) \end{cases}$$

(ii) $\ell = 1$

$$b_{1,p} = \frac{|S^{q-1}|^{1/2}}{q^{1/2}} \langle \nabla\Phi_k(\xi), e_p(\xi)\rangle \epsilon^q (1 + O(\epsilon)) \times \begin{cases} -1/(q-1) & (\tau > -1) \\ (\sigma - 1)/(q - 1 + \sigma) & (\tau = -1) \\ 1 & (\tau < -1) \end{cases}$$

Lemma. For any $N \in \mathbb{N}$, there exists $d_N > 0$ (independent of $\xi \in M$) such that

$$|b_{\ell,p}| \leqq \frac{d_N}{\gamma(\ell)^N} \begin{cases} \epsilon^{\ell+q} & (\tau \geqq 0) \\ \epsilon^{\ell+q+\tau} & (-1 < \tau < 0) \\ \epsilon^{\ell+q-1} & (\tau \leqq -1) \end{cases}, \quad (1 \leqq p \leqq \iota(\ell), \ell \geqq 2).$$

Proof for the theorems

For any sequence of positive values $\{\epsilon_p\}_{p=1}^{\infty}$ with $\lim_{p\to\infty}\epsilon_p = 0$, there exists a subsequence $\{\zeta_p\}_{p=1}^{\infty}$ and orthonormal systems of eigenfunctions $\{\Phi'_k\}_{k=1}^{\infty}$ and $\{\Phi''_k\}_{k=1}^{\infty}$ of (1) corresponding to $\{\lambda_k\}_{k=1}^{\infty}$, respectively such that

$$(\Phi'_k, \Phi'_\ell)_{L^2(\Omega)} = \delta(k,\ell), \quad (\Phi''_k, \Phi''_\ell)_{L^2(\Omega)} = \delta(k,\ell) \quad (k,\ell \in \mathbb{N}),$$

$$\lim_{p\to\infty} \|\Phi^R_{k,\zeta_p} - \Phi'_k\|_{L^2(\Omega(\zeta_p))} = 0, \quad \lim_{p\to\infty} \|\Phi^N_{k,\zeta_p} - \Phi''_k\|_{L^2(\Omega(\zeta_p))} = 0.$$

Calculation of the limit behavior of $\lambda_k^R(\epsilon) - \lambda_k$.

$$\int_{\Omega(\epsilon)} (\Delta \Phi_{k,\epsilon}^R + \lambda_k^R(\epsilon)\Phi_{k,\epsilon}^R)\widetilde{\Phi}_{k,\epsilon} dx = 0 \tag{7}$$

Assume the situation $\Phi_{k,\epsilon}^R \longrightarrow \Phi_k'$ for $\epsilon = \zeta_p \to 0$ as in Proposition 2.
Calculation on the above integral relation gives

$$(\lambda_k(\epsilon) - \lambda_k)\int_{\Omega(\epsilon)} \Phi_k(x)\Phi_{k,\epsilon}(x)dx = I_1(\epsilon) + I_2(\epsilon) + I_3(\epsilon) + I_4(\epsilon). \tag{8}$$

where

$$I_1(\epsilon) = -\int_{\Gamma(M,r_0)} \frac{\partial G_{k,\epsilon}}{\partial \nu_1}(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dS$$

$$I_2(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} G_{k,\epsilon}(x)(\Delta\Phi_k'(x) + \lambda_k(\epsilon)\Phi_{k,\epsilon}(x))dx$$

$$I_3(\epsilon) = \int_{B(M,r_0)\setminus B(M,\epsilon)} (\Delta G_{k,\epsilon}(x))(\Phi_{k,\epsilon}(x) - \Phi_k'(x))dx$$

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left(\frac{\partial G_{k,\epsilon}}{\partial \nu_1} \Phi_k' - G_{k,\epsilon} \frac{\partial \Phi_k'}{\partial \nu_1} \right) dS$$

$I_4(\epsilon)$ is also written

$$I_4(\epsilon) = \int_{\Gamma(M,\epsilon)} \left((\frac{\partial \Phi_k}{\partial \nu_1} + \sigma \epsilon^\tau \Phi_k) \Phi_k' - G_{k,\epsilon} (\frac{\partial \Phi_k'}{\partial \nu_1} + \sigma \epsilon^\tau \Phi_k') \right) dS.$$

Careful evaluation on I_1, I_2, I_3, I_4 gives the perturbation formula in Theorem.

(B) Domain with partial degeneration

$D \subset \mathbb{R}^n$: a bounded domain (or a finite union of bounded domains) with a smooth boundary. The perturbed domain

$$\Omega(\zeta) = D \cup Q(\zeta) \subset \mathbb{R}^n$$

Here $Q(\zeta)$ is a thin set which approaches a lower dimensional set L as $\zeta \to 0$.

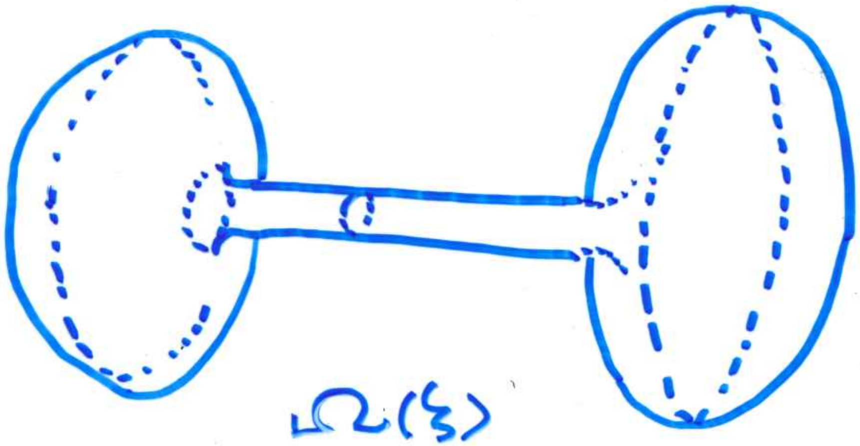
Eigenvalue problem

$$\Delta\Phi + \mu\Phi = 0 \quad \text{in} \quad \Omega(\zeta), \qquad \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega(\zeta) \tag{9}$$

Let $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ be the eigenvalues with the corresponding eigenfunctions $\Phi_{k,\zeta}$ $(k \geqq 1)$ such that

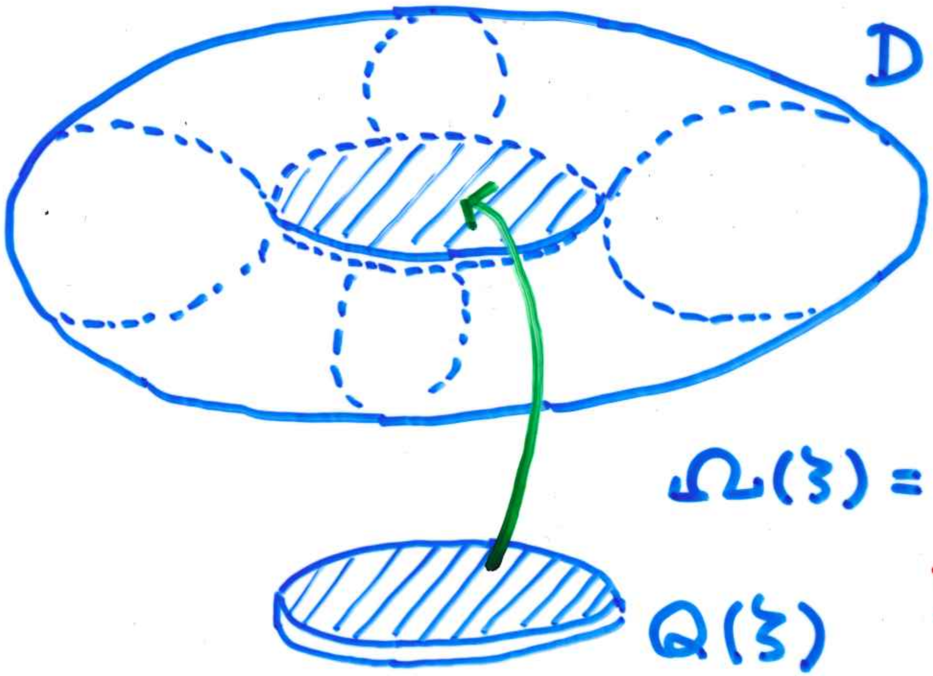
$$(\Phi_{k,\zeta}, \Phi_{k',\zeta})_{L^2(\Omega(\zeta))} = \delta(k, k') \quad (k, k' \geqq 1) \quad \text{(Kronecker's delta)}$$

Example



$n = 3$
$\ell = 1$
$m = 2$

Dumbbell



$n = 3$
$\ell = 2$
$m = 1$

$\Omega(\xi) = D \cup Q(\xi)$

Doughnut + Pancake

Basic question : What is the limitting behavior of $\mu_k(\zeta)$ for $\zeta \to 0$?

For the Dumbbell domain, there are results. Beale('73), Fang('93), Jimbo('93), Gadylshin('93), Arrieta('95), Jimbo-Morita('95), Anné('87),...

Convergence of the eigenvalues. Perturbation formula (first order approximation) is studied.

In this lecture I deal with more genral cases. Hereafter I mainly talk about the results in Jimbo-Kosugi('09).

The construction of $\Omega(\zeta) = D \cup Q(\zeta)$

Let $n, \ell, m \in \mathbb{N}$ with $n = \ell + m$. $\quad x = (x', x'') \in \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$
$D \subset \mathbb{R}^n$, $L \subset \mathbb{R}^\ell$ bounded domains (finite disjoint union of bounded domains) with smooth boundaries.

Some symbols:

$$B^{(m)}(s) := \{x'' \in \mathbb{R}^m \mid |x''| < s\}, \quad L(s) := \{x' \in L \mid \mathrm{dist}(x', \partial L) > s\}$$

Assumption: There exists $\zeta_0 > 0$ such that

$$(\overline{L} \times B^{(m)}(3\zeta_0)) \cap D = \partial L \times B^{(m)}(3\zeta_0) \subset \partial D$$

There exists a function $\rho = \rho(t) \in C^3((-\infty, 0)) \cap C^0((-\infty, 0])$ such that

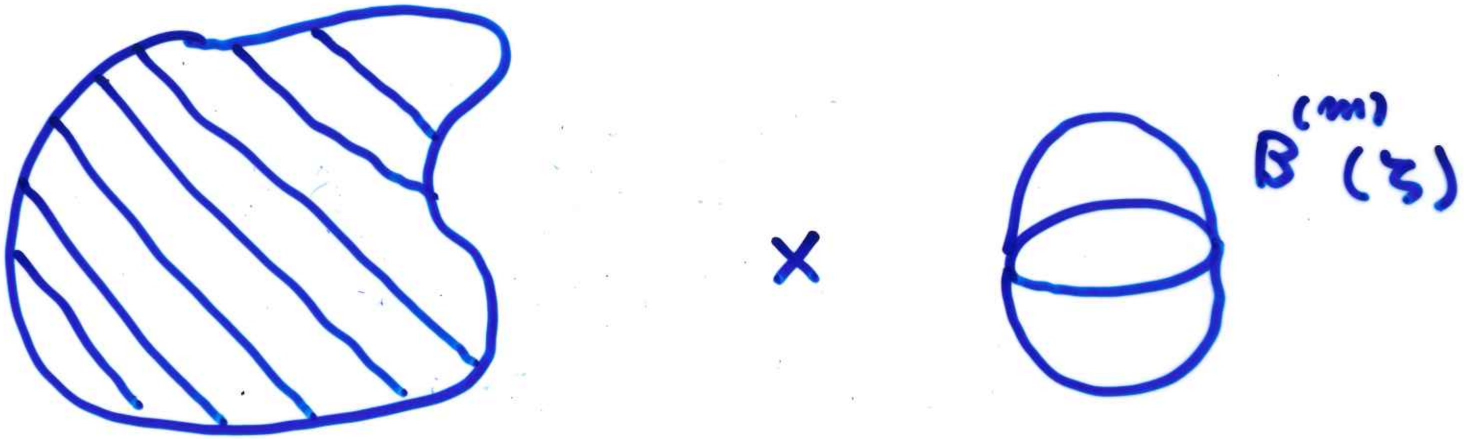
$$\rho(t) = 1 \ (t \leqq -1), \ \rho'(t) > 0 \ (-1 < t < 0), \ \rho(0) = 2, \ \lim_{s \uparrow 2} d^k \rho^{-1}(s)/ds^k = 0 \quad (1 \leqq k \leqq 3)$$

Put $Q(\zeta) = Q_1(\zeta) \cup Q_2(\zeta)$ where $Q_1(\zeta) = L(2\zeta) \times B^{(m)}(\zeta)$ and

$$Q_2(\zeta) = \{(\xi + s\nu'(\xi), \eta) \mid \mathbb{R}^\ell \times \mathbb{R}^m \mid -2\zeta \leqq s \leqq 0, \xi \in \partial L, |\eta| < \zeta\rho(s/\zeta)\}$$

For general ℓ, m

$$Q(\zeta) \doteqdot L \times B^{(m)}(\zeta) \subset \mathbb{R}^{\ell} \times \mathbb{R}^{m} = \mathbb{R}^{n}$$



ℓ-dim domain m-dim small ball

To express the limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ we prepare the notation.

Definition. $\{\omega_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

$$\Delta\phi + \omega\phi = 0 \text{ in } D, \quad \partial\phi/\partial\nu = 0 \text{ on } \partial D \tag{10}$$

Definition. $\{\lambda_k\}_{k=1}^{\infty}$ is the system of eigenvalues of

$$\Delta'\psi + \lambda\psi = 0 \text{ in } L, \quad \psi = 0 \text{ on } \partial L \tag{11}$$

where

$$\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_\ell^2}$$

The limit of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ is given by the following result.

Proposition. $\lim_{\zeta\to 0}\mu_k(\zeta) = \mu_k$ for any $k \geqq 1$ where $\{\mu_k\}_{k=1}^{\infty}$ is given by rearranging $\{\omega_k\}_{k=1}^{\infty} \cup \{\lambda_k\}_{k=1}^{\infty}$ in increasing order with counting multiplicity.

Remark. μ_k is written as

$$\mu_k = \max_{1\leqq j\leqq k}\left(\min(\omega_{k+1-j}, \lambda_j)\right).$$

Classification of eigenvalues

Definition

$$E_I = \{\omega_k\}_{k=1}^{\infty} \setminus \{\lambda_k\}_{k=1}^{\infty}, \qquad E_{II} = \{\lambda_k\}_{k=1}^{\infty} \setminus \{\omega_k\}_{k=1}^{\infty}, \qquad E_{III} = \{\omega_k\}_{k=1}^{\infty} \cap \{\lambda_k\}_{k=1}^{\infty}$$

Relation to the eigenfunctions

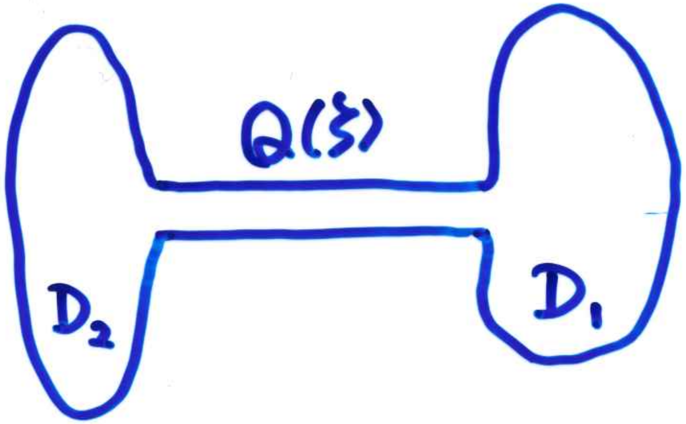
Let $\{\Phi_{k,\zeta}\}_{k=1}^{\infty} \subset L^2(\Omega(\zeta))$ be the (complete) orthonormal system corresponding to $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ of (9).

Proposition.

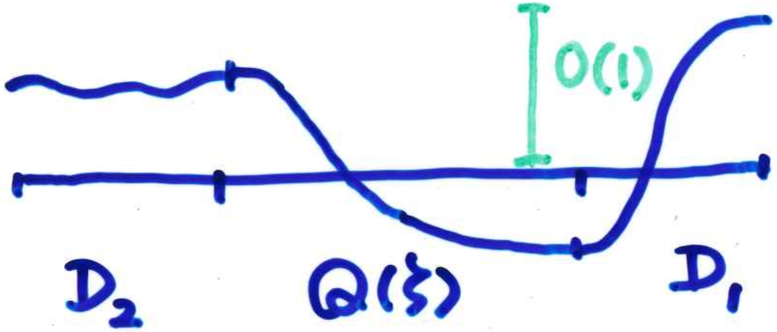
$$\lim_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} = 0 \Longleftrightarrow \mu_k \in E_I$$

$$\lim_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} = 0 \Longleftrightarrow \mu_k \in E_{II}$$

$$\liminf_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(Q(\zeta))} > 0, \; \liminf_{\zeta \to 0} \|\Phi_{k,\zeta}\|_{L^2(D)} > 0 \Longleftrightarrow \mu_k \in E_{III}$$



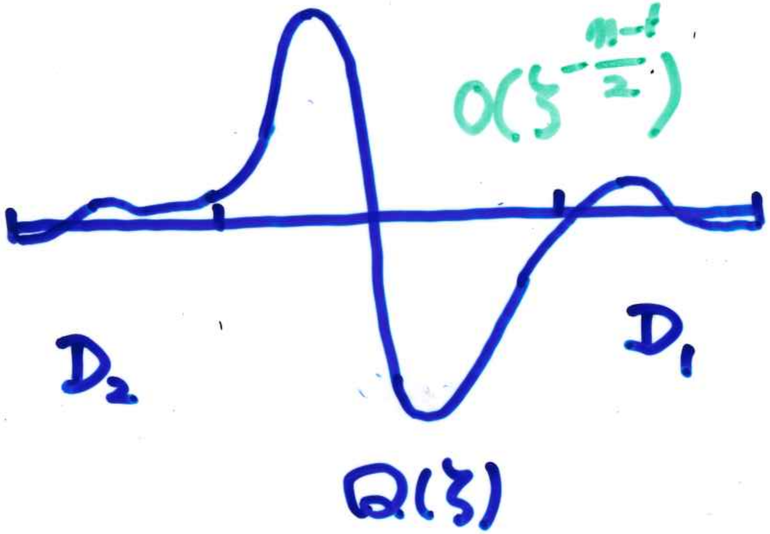
$\{\mu_k(\xi)\}_{k=1}^{\infty} \sim \{\omega_k\}_{k=1}^{\infty} \cup \{\lambda_k\}_{k=1}^{\infty}$



$\Phi_{k,\xi}$

Type I

$\{\omega_k\}_{k=1}^{\infty} \setminus \{\lambda_k\}_{k=1}^{\infty}$



$\Phi_{k,\xi}$

Type II

$\{\lambda_k\}_{k=1}^{\infty} \setminus \{\omega_k\}_{k=1}^{\infty}$

Type III $\{\omega_k\} \cap \{\lambda_k\}$

$\Phi_{k,\xi}(x) \sim$

Linear combination of two types

Proposition (Convergence rate)

$$\mu_k \in E_I \Longrightarrow \mu_k(\zeta) - \mu_k = O(\zeta^m)$$

$$\mu_k \in E_{II} \Longrightarrow \mu_k(\zeta) - \mu_k = \begin{cases} O(\zeta) & (m \geqq 2) \\ O(\zeta \log(1/\zeta)) & (m = 1) \end{cases}$$

For $\mu_k \in E_{III}$, a mixed situation occurs (as seen later).

Some preparation(uniform converegence)

Consider the following semilinear elliptic equation in $\Omega(\zeta)$.

$$\Delta u + f_\zeta(u) = 0 \quad \text{in} \quad \Omega(\zeta), \quad \partial u/\partial \nu = 0 \quad \text{on}$$

Here $\zeta > 0$ is a parameter and the nonlinear term $f_\zeta(u)$ is assumed to be a C^1 function in $\mathbb{R}$ such that $(\partial f_\zeta/\partial u)(u)$ is uniformly bounded in $\mathbb{R}$ and $f_\zeta(u)$ converges locally uniformly to a C^1 function $f_0(u)$ for $\zeta \to 0$.

Theorem. Let $\{\zeta_p\}_{p=1}^\infty$ be a positive sequence which converges to 0 as $p \to \infty$ and let $u_{\zeta_p} \in C^2(\overline{\Omega(\zeta_p)})$ be a solution of the above equation for $\zeta = \zeta_p$ such that

$$\sup_{p \geqq 1} \sup_{x \in \Omega(\zeta_p)} |u_{\zeta_p}(x)| < \infty.$$

Then there exists a subsequence $\{\sigma_p\}_{p=1}^\infty \subset \{\zeta_p\}_{p=1}^\infty$ and functions $w \in C^2(\overline{D})$ and $V \in C^2(\overline{L})$ such that

$$\Delta w + f_0(w) = 0 \quad \text{in} \quad D,\ \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial D \quad \text{(Neumann B.C.)},$$
$$\Delta' V + f_0(V) = 0 \quad \text{in} \quad L,\ V(x') = w(x', o'') \quad \text{for} \quad x' \in \partial L,$$
$$\lim_{p\to\infty} \sup_{x\in D} |u_{\sigma_p}(x) - w(x)| = 0,$$
$$\lim_{p\to\infty} \sup_{(x',x'')\in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - V(x')| = 0,$$

where $\Delta' = \sum_{k=1}^{\ell} \partial^2/\partial x_k^2$. Note that $\partial L \times \{o''\} \subset \partial D$.

Perturbation formula [Type (I)]

Let $\{\phi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (10) (eigenvalue problem in D) orthonormalized in $L^2(D)$.

Assume $\mu_k \in E_I$ and there exists $k' \in \mathbb{N}$ such that $\mu_k = \omega_{k'}$. Assume also that $\omega_{k'}$ is a simple eigenvalue of (10).

Theorem.

$$\mu_k(\zeta) - \mu_k = S(m)\alpha(k)\zeta^m + o(\zeta^m)$$

where

$$\alpha(k) = \int_{\partial L} \frac{\partial V_{k'}}{\partial \nu'}(\xi)\phi_{k'}(\xi, o'')dS'$$

$V_{k'}(x')$ is the unique solution $V \in C^2(\overline{L})$ of

$$\Delta' V + \omega_{k'} V = 0 \text{ in } L, \quad V(\xi) = \phi_{k'}(\xi, o'') \text{ for } \xi \in \partial L.$$

$S(m)$ is the m-dimensional volume of the unit ball in $\mathbb{R}^m$.

Perturbation formula [Type (II)]

Let $\{\psi_k\}_{k=1}^{\infty}$ be the system of eigenfunctions of (11) (eigenvalue problem in L) orthonormalized in $L^2(L)$.

Assume $\mu_k \in E_{II}$ and there exists $k'' \in \mathbb{N}$ such that $\mu_k = \lambda_{k''}$. Assume also that $\lambda_{k''}$ is a simple eigenvalue of (11).

Theorem.

$$\mu_k(\zeta) - \mu_k = -\frac{2}{\pi}\beta(k'')\zeta \log(1/\zeta) + o(\zeta \log(1/\zeta)) \quad (m = 1),$$

$$\mu_k(\zeta) - \mu_k = -T(\rho, m)\beta(k'')\zeta + o(\zeta) \quad (m \geqq 2).$$

where

$$\beta(k'') = \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \right)^2 dS'$$

and $T(\rho, m)$ is the number which depends on $\Omega(\zeta)$ (to be explained later).

Remark. For the case of Dumbbell, Gadylshin('93) obtained this result $m = 2$ and Arrieta ('95) obtained this result for $m = 1$.

Quantity $T(\rho, m)$ $(m \geqq 2)$

Harmonic function G in the set $H = H_1 \cup H_2 \subset \mathbb{R} \times \mathbb{R}^m$ where H_1, H_2 are given

$$H_1 = (0, \infty) \times \mathbb{R}^m, \qquad H_2 = \{(s, \eta) \in \mathbb{R} \times \mathbb{R}^m \mid |\eta| < \rho(s),\ s \leqq 0\}.$$

Proposition. There exists a solution G to

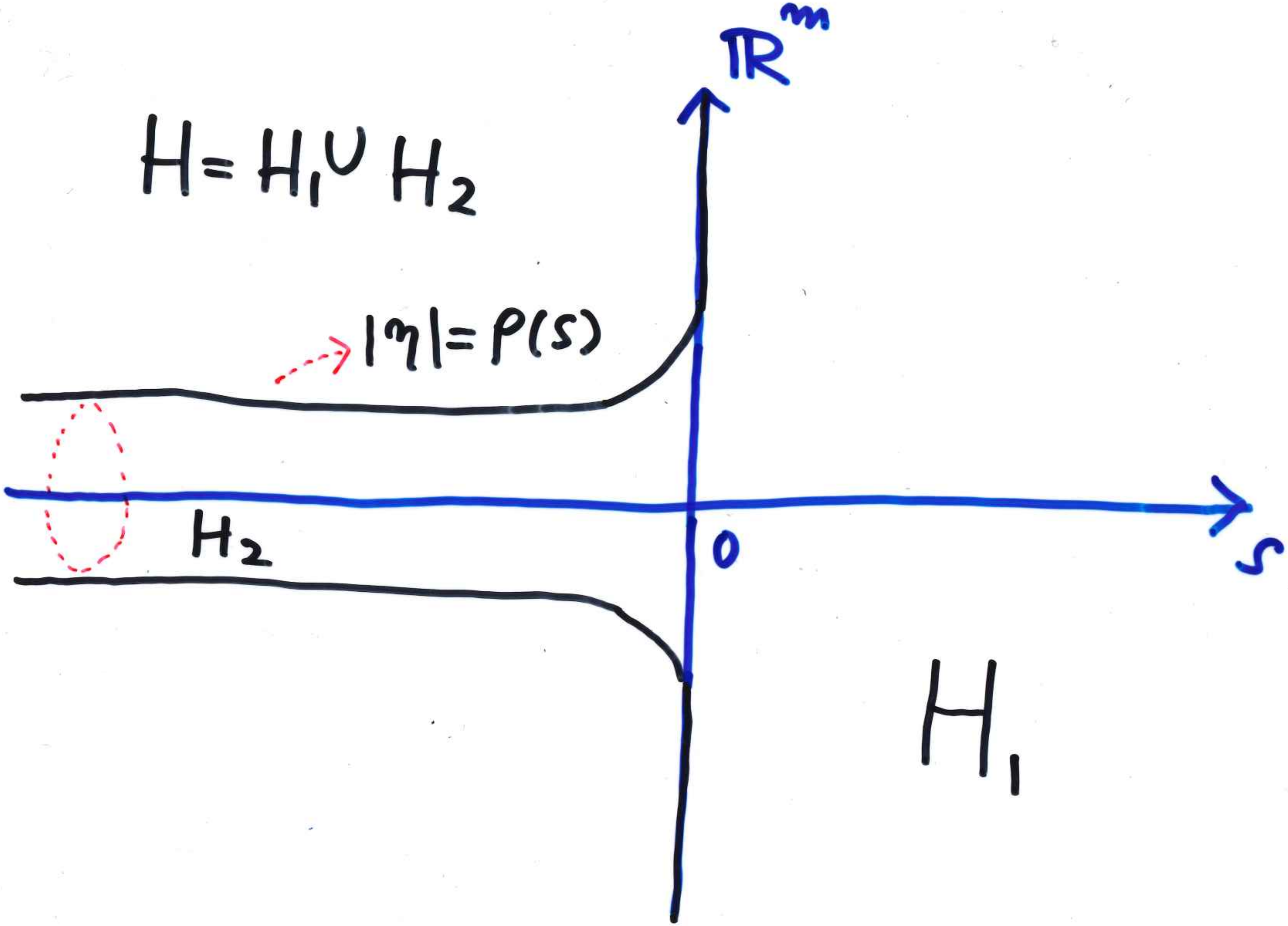
$$\frac{\partial^2 G}{\partial s^2} + \sum_{j=1}^{m} \frac{\partial^2 G}{\partial \eta_j^2} = 0 \quad ((s, \eta) \in H) \qquad \frac{\partial G}{\partial \boldsymbol{n}} = 0 \quad ((s, \eta) \in \partial H)$$

such that

$$G(z) = G(s, \eta) \longrightarrow 0 \quad \text{for} \quad (z \in H_1, |z| \to \infty)$$
$$G(s, \eta) - (-\kappa_1 s + \kappa_2) \longrightarrow 0 \quad \text{for} \quad (z \in H_2, |z| \to \infty)$$

where $\kappa_1 > 0, \kappa_2$ are real constants. κ_2/κ_1 is uniquely determined by H.

Definition. $T(\rho, m) = \kappa_2/\kappa_1$.



$\mathbb{R}^m$
$H = H_1 \cup H_2$
$|\eta| = \rho(s)$
H_2
0
s
H_1

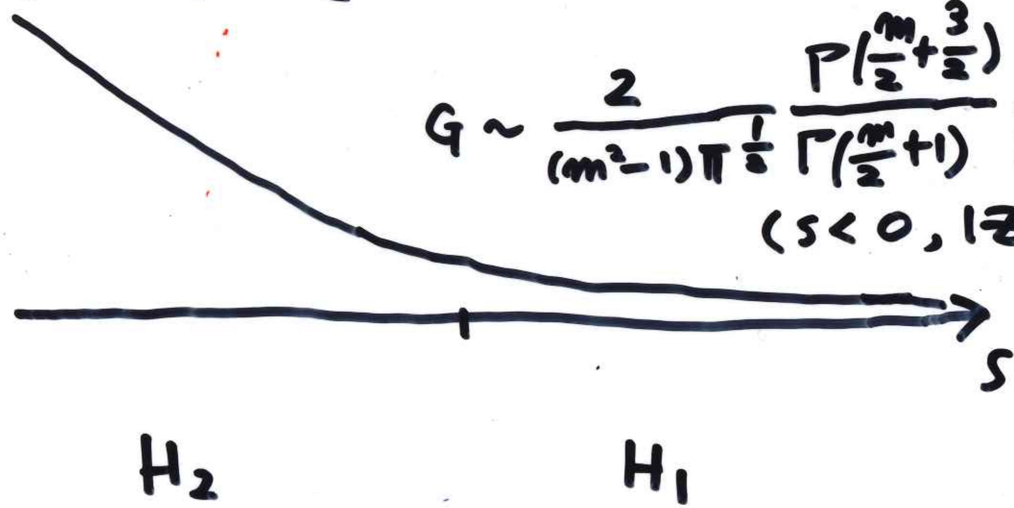
$$z = (s, \eta) \in \mathbb{R} \times \mathbb{R}^m$$

$m \geqq 2$

$G \sim -k_1 s + k_2 \quad (s \to -\infty)$

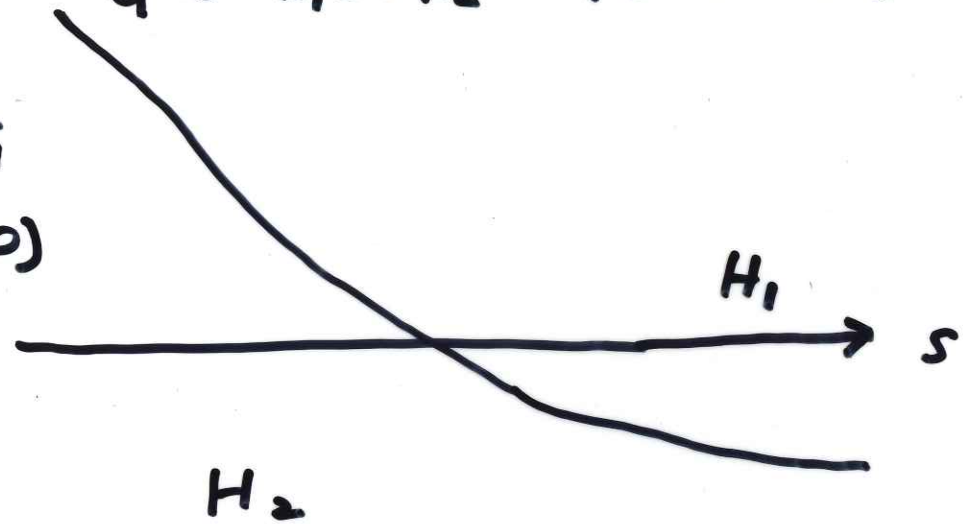
$$G \sim \frac{2}{(m^2-1)\pi^{\frac{1}{2}}} \frac{\Gamma\left(\frac{m}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)} \frac{1}{|z|^{m-1}}$$

$(s < 0, |z| \to \infty)$



$m = 1$

$G \sim -k_1 s + k_2 \quad (s \to -\infty)$



$$G \sim \frac{2k_1}{\pi} \log \frac{1}{|z|}$$

$(s < 0, |z| \to \infty)$

Perturbation formula [Type (III)]

Assume $\mu_k \in E_{III}$ and there exists $k', k'' \in \mathbb{N}$ such that $\mu_k = \omega_{k'} = \lambda_{k''}$. Assume also that $\omega_{k'}$ is simple eigenvalue of (10) and $\lambda_{k''}$ is a simple eigenvalue of (11).

We have the situation

$$\mu_{k-1} < \mu_k = \mu_{k+1} < \mu_{k+2}.$$

Theorem. For $m = 1$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_1^-(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_1^+(k', k'')\zeta^{1/2} + o(\zeta^{1/2})$$

where $\gamma_1^{\pm}(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{2}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & 0 \end{pmatrix}$$

Theorem. For $m = 2$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_2^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_2^+(k', k'')\zeta + o(\zeta)$$

where $\gamma_2^\pm(k', k'')$ are eigenvalues of

$$\begin{pmatrix} 0 & \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' \\ \sqrt{\pi}\int_{\partial L}(\partial\psi_{k''}/\partial\nu')(\xi)\phi_{k'}(\xi, o'')dS' & -T(\rho, 2)\int_{\partial L}((\partial\psi_{k''}/\partial\nu')(\xi))^2 dS' \end{pmatrix}$$

Remark. For the case of Dumbbell ($m = 2, n = 3$), Gadylshin ('05) got this result. See Jimbo-Kosugi('09) for more genral cases.

Theorem. Assume $T(\rho, m) > 0$. For $m \geqq 3$, we have

$$\mu_k(\zeta) - \mu_k = \gamma_m^-(k', k'')\zeta + o(\zeta)$$
$$\mu_{k+1}(\zeta) - \mu_{k+1} = \gamma_m^+(k', k'')\zeta^{m-1} + o(\zeta)$$

where

$$\gamma_m^-(k', k'') = -T(\rho, m) \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'}(\xi) \right)^2 dS'$$

$$\gamma_m^+(k', k'') = S(m)T(\rho, m)^{-1} \left(\int_{\partial L} (\frac{\partial \psi_{k''}}{\partial \nu'}(\xi))^2 dS' \right)^{-1} \left(\int_{\partial L} \frac{\partial \psi_{k''}}{\partial \nu'}(\xi)\phi_{k'}(\xi, o'')dS' \right)^2$$

In the case $T(\rho, m) < 0$, the right hand sides are exchanged.

Approximate eigenfunction $\tilde{\varphi}_{k',\zeta}$, $\tilde{\psi}_{k'',\zeta}$

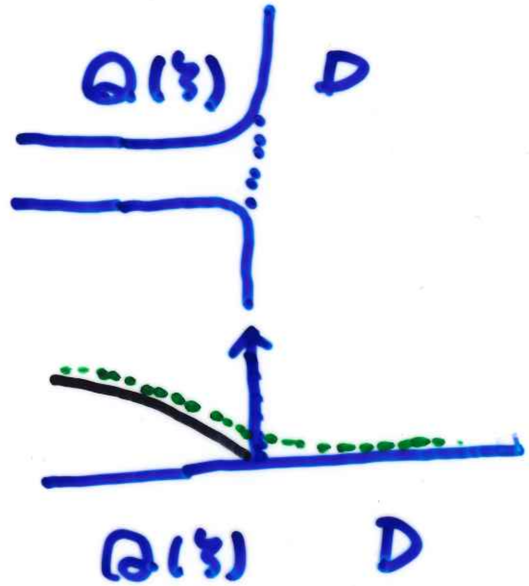
$(\omega_{k'}, \varphi_{k'})$ of (10)

$$\tilde{\varphi}_{k',\zeta}(x) = \begin{cases} \varphi_{k'}(x) & x \in D \setminus \Sigma^+(2\zeta) \\ N_{k',\zeta}(x) & x \in \Sigma^-(\zeta, 2\zeta) \cup \Sigma^+(2\zeta) \\ V_{k'}(x') & x = (x', x'') \in Q(\zeta) \setminus \Sigma^-(\zeta, 2\zeta) \end{cases}$$

$(k' \geq 1)$

$(\lambda_{k''}, \psi_{k''})$ of (11)

$$\tilde{\psi}_{k'',\zeta}(x) = \tilde{\psi}_{k''}(x) + \zeta\, v_{k'',\zeta}(x)$$



Rayleigh Quotient

$$R_\xi[\Phi] = \frac{\int_{\Omega(\xi)} |\nabla \Phi|^2 dx}{\int_{\Omega(\xi)} |\Phi|^2 dx}$$

Rough observation case $m = 1$

$$\Phi(x) = X \tilde{\varphi}_{k',\xi}(x) + Y \tilde{\varphi}_{k'',\xi}(x) \left(\frac{1}{2\xi}\right)^{\frac{1}{2}}$$

$$(X, Y \in \mathbb{R})$$

$$R_\xi[\Phi] = \frac{\left(A_\xi \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}\right)}{\left(B_\xi \begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}\right)}$$

$$\Longleftrightarrow \quad (A_\zeta - \mu B_\zeta)\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A_\zeta = \begin{pmatrix} \omega_{k'} & 0 \\ 0 & \lambda_{k''} \end{pmatrix} + \zeta \begin{pmatrix} a_{1,\zeta} & 0 \\ 0 & 0 \end{pmatrix} + \zeta^{\frac{1}{2}} \begin{pmatrix} 0 & a_{2,\zeta} \\ a_{2,\zeta} & 0 \end{pmatrix} + \zeta \log\frac{1}{\zeta} \begin{pmatrix} 0 & 0 \\ 0 & a_{3,\zeta} \end{pmatrix}$$

$$B_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \zeta \begin{pmatrix} b_{1,\zeta} & 0 \\ 0 & 0 \end{pmatrix} + \zeta^{\frac{1}{2}} \begin{pmatrix} 0 & b_{2,\zeta} \\ b_{2,\zeta} & 0 \end{pmatrix} + \zeta \log\frac{1}{\zeta} \begin{pmatrix} 0 & 0 \\ 0 & b_{3,\zeta} \end{pmatrix}$$

$$a_{1,\zeta} = 2\int_L |\nabla' V_{k'}|^2 \, dx' + o(1), \quad b_{1,\zeta} = 2\int_L V_{k'}^2 \, dx' + o(1)$$

$$a_{2,\zeta} = \sqrt{2}\int_L \nabla' V_{k'} \nabla' \psi_{k''} \, dx' + \frac{1}{\sqrt{2}} \omega_{k'} \int_D \phi_{k'} v_{k''} \, dx + o(1)$$

$$b_{2,\zeta} = \sqrt{2}\int_L V_{k'} \psi_{k''} \, dx' + \frac{1}{\sqrt{2}} \int_D \phi_{k'} v_{k''} \, dx + o(1)$$

$$a_{3,\xi} = \frac{2}{\pi} \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'} \right)^2 ds' + o(1)$$

$$b_{3,\xi} = \frac{4}{\pi \lambda_{k''}} \int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu'} \right)^2 ds' + o(1)$$

Eigenvalue problem (2x2 matrix)

$$\begin{pmatrix} \omega_{k'} - \mu & 0 \\ 0 & \lambda_{k''} - \mu \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \xi (a_{1,\xi} - \mu b_{1,\xi}) & \xi^{\frac{1}{2}} (a_{2,\xi} - \mu b_{2,\xi}) \\ \xi^{\frac{1}{2}} (a_{2,\xi} - \mu b_{2,\xi}) & \xi \log \frac{1}{\xi} (a_{3,\xi} - \mu b_{3,\xi}) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

what is the behaviors of 2 eigenvalues for $\xi \downarrow 0$

2 cases $\omega_{k'} \neq \lambda_{k''}$, $\omega_{k'} = \lambda_{k''}$

$\omega_{k'} \neq \lambda_{k''}$ case

① Assume μ is close to $\omega_{k'}$

$$\mu - \omega_{k'} = (a_{1,\xi} - \mu b_{1,\xi})\xi + o(\xi)$$
$$= 2\int_L (|\nabla' V_{k'}|^2 - \omega_{k'} V_{k'}^2)\, dx' \cdot \xi + o(\xi)$$

② Assume μ is close to $\lambda_{k''}$

$$\mu - \lambda_{k''} = \xi \log\frac{1}{\xi}\,(a_{3,\xi} - \mu b_{3,\xi}) + o(\xi \log\frac{1}{\xi})$$
$$= -\frac{2}{\pi}\int_{\partial L} \left(\frac{\partial \psi_{k''}}{\partial \nu}\right)^2 ds \cdot \xi \log\frac{1}{\xi} + o(\xi \log\frac{1}{\xi})$$

$\omega_{k'} = \lambda_{k''}$ case

$$\begin{array}{c} \omega_{k'} \\ \parallel \\ \mu_k \end{array}$$

$$(\mu_k - \mu)\begin{pmatrix} X \\ Y \end{pmatrix} + \zeta^{\frac{1}{2}}\begin{pmatrix} \zeta^{\frac{1}{2}}(a_{1,\zeta} - \mu b_{1,\zeta}) , & (a_{2,\zeta} - \mu b_{2,\zeta}) \\ (a_{2,\zeta} - \mu b_{2,\zeta}) , & \zeta^{\frac{1}{2}} \log\frac{1}{\zeta} (a_{3,\zeta} - \mu b_{3,\zeta}) \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mu_k - \mu = \pm (a_2 - \mu_k b_{2,\zeta}) \zeta^{\frac{1}{2}} + o(\zeta^{\frac{1}{2}})$$

$$= \pm\sqrt{2}\int_{\partial L} \phi_{k'}(\zeta, 0'') \frac{\partial \psi_{k''}}{\partial \nu'}(\zeta)\, ds' \zeta^{\frac{1}{2}} + o(\zeta^{\frac{1}{2}})$$

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